# Holomorphic Dynamics on Bounded Symmetric Domains of Finite Rank

Jeffrey Michael Rigby

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*Nullius in verba*

## Statement of Originality

I, Jeffrey Michael Rigby, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

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**Signature:** M Rigby **Date:** 10th March 2015

**Details of collaboration and publications:**

A significant proportion of Sections 3.2 and 3.3 has been published in the following paper, written in collaboration with my supervisor Professor Cho-Ho Chu:

- C.-H. Chu and M. Rigby, Iteration of self-maps on a product of Hilbert balls, Journal of Mathematical Analysis and Applications **411** (2014), no. 2, 773 – 786.

Apart from Proposition 3.2.5, Theorem 3.3.1 and Proposition 3.3.4, which were obtained jointly with my supervisor, the other results in these sections are my contribution.

# Acknowledgements

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# Abstract

In this thesis we present new results in holomorphic dynamics on rank-2 bounded symmetric domains, which can be infinite-dimensional. Some of these results have been published in [12]. Together with other current research, this establishes a comprehensive theory of the dynamics of fixed-point-free holomorphic self-maps on rank-2 bounded symmetric domains. Jordan theory is the novel approach used to achieve these results, which relates to the hyperbolic geometry of bounded symmetric domains.

We examine the iterates of a fixed-point-free holomorphic self-map on the open unit balls $D$ of two classes of JB*-triples:

1. A finite $\ell_\infty$-sum $V$ of Hilbert spaces;
2. The Banach space $L(\mathbb{C}^2, H)$ of all bounded linear operators from $\mathbb{C}^2$ to a Hilbert space $H$.

The main results in each case are an explicit description in Jordan theoretic terms of the invariant domains of $f$ and an analysis of the subsequential limit points of the iterates of $f$ in the topology of locally uniform convergence. Details are given as follows.

Let $f : D \to D$ be a compact fixed-point-free holomorphic map. We show the existence of horospheres $S(\xi, \lambda)$ at a boundary point $\xi$ of $D$, parameterised by a positive number $\lambda$, satisfying $f(S(\xi, \lambda) \cap D) \subset S(\xi, \lambda) \cap D$. These horospheres are described in terms of the Bergmann operator.

In Case 1, where $V$ is a sum of $p$ Hilbert spaces $V_1, \ldots, V_p$, the horosphere $S(\xi, \lambda)$ at the boundary point $\xi = (\xi_1, \ldots, \xi_p)$ has the form

$$S(\xi, \lambda) = \prod_{j=1}^{p} S_j(\xi_j, \lambda)$$

where, for some nonempty subset $J$ of $\{1, ..., p\}$, $S_j(\xi_j, \lambda) = \overline{D_j}$ for $j \notin J$ and, for $j \in J$,

$$S_j(\xi_j, \lambda) = \lambda_j^2 \xi_j + B(\lambda_j \xi_j, \lambda_j \xi_j)^{1/2}(\overline{D}_j)$$

where $D_j$ is the open unit ball of $V_j$ and $\lambda_j > 0$.

In Case 2, the horosphere has the form

$$S(\xi, \lambda) = \lambda_1^2 e + \lambda_2^2 v + B\left(\lambda_1 e + \lambda_2 v, \lambda_1 e + \lambda_2 v\right)^{1/2}(\overline{D})$$

where $\lambda_1 \in (0, 1)$, $\lambda_2 \in [0, 1)$ and $e$ is a minimal tripotent.

Leveraging these results we analyse the subsequential limit points of $(f^n)$. In Case 1, we prove that each limit point $h$ of the iterates $(f^n)$ satisfies $\xi_j \in \overline{\pi_j \circ h(D)}$ for all $j \in J$ and $\pi_j \circ h(\cdot) = \xi_j$ whenever $\pi_j \circ h(D)$ meets the boundary of $D_j$, where $\pi_j$ is the coordinate map $(x_1, ..., x_p) \in \overline{D} \mapsto x_j \in \overline{D_j}$.

In Case 2, the boundary point $\xi$, takes the form $e + \beta v$, where $e$ is a minimal tripotent, $\beta \in [0, 1]$ and, if $\beta \neq 0$, $v$ is a minimal tripotent. For each limit point $h$ of $(f^n)$, we have $h(D) \subset K_u$ for some tripotent $u$ satisfying $\overline{K_u} \cap \overline{K_e} \neq \emptyset$, where $K_a$ denotes the boundary component in $\overline{D}$ containing $a$.

# Contents

# Introduction

In this thesis we study holomorphic dynamics on infinite-dimensional bounded symmetric domains of rank-2. The finite-dimensional rank-2 bounded symmetric domains are the Cartesian product of two Euclidean balls, and the open unit balls of $M_{m2}(\mathbb{C})$, $S_5(\mathbb{C})$, $H_2(\mathbb{C})$, the finite-dimensional spin factors and $M_{12}(\mathcal{O})$, where $M_{m2}(\mathbb{C})$ denotes the $m \times 2$ complex matrices with $m \geq 2$, $S_5(\mathbb{C})$ represents the $5 \times 5$ skew-symmetric complex matrices, $H_2(\mathbb{C})$ denotes the $2 \times 2$ symmetric complex matrices, $M_{12}(\mathcal{O})$ represents the $1 \times 2$ complex octonian matrices, and we will define a spin factor of any dimension in Section 4.1. Holomorphic dynamics on the bidisc is well understood by the seminal work of Hervé [20], and the dynamics on the other finite-dimensional domains can be treated as a special case of the finite-dimensional theory which has been developed by many authors [1, 7, 8, 33, 34, 35 and 36].

The infinite-dimensional bounded symmetric domains of rank-2 consist of the open unit balls of infinite-dimensional spin factors, of a product of two Hilbert spaces which is infinite-dimensional, and of the space $L(\mathbb{C}^2, H)$ of bounded linear operators from $\mathbb{C}^2$ to an infinite-dimensional Hilbert space $H$. Together with the work of [10] for the open unit ball of spin factor, known as a Lie ball, a comprehensive theory of the dynamics of a fixed-point-free holomorphic map on infinite-dimensional rank-2 domains is established. This, alongside the aforementioned finite-dimensional theory, provides an extensive picture of the dynamics of a holomorphic map on all rank-2 bounded symmetric domains.

We begin in Chapter 1 with holomorphic dynamics in one dimension, focusing on the two renowned theorems, the Wolff Theorem (*q.v.* Theorem 1.1.1) and the Denjoy-Wolff Theorem (*q.v.* Theorem 1.1.3), which we present from our viewpoint of hyperbolic geometry and notation akin to that which will be used later. In Section 1.2 we introduce bounded symmetric domains, including Élie Cartan's famous classification [6] of the finite-dimensional bounded symmetric domains, and the notion of *boundary components* of a domain. We conclude the chapter with a review of the topology of locally uniform convergence, as a backdrop for the infinite-dimensional holomorphic dynamics we study in this thesis.

In Chapter 2, we discuss the Jordan algebraic structures vital to our subsequent investigation of holomorphic dynamics on bounded symmetric domains. This begins with the important concept of a JB*-triple. We discuss some examples in Sections 2.1 and 2.2. The Jordan algebraic classification of bounded symmetric domains is explained in Section 2.3.

We initiate our investigation in Chapter 3 with a study of the dynamics of a holomorphic self-map on a product of Hilbert balls. These domains are a natural generalisation of the finite-dimensional polydiscs. Therefore, in Section 3.1, we motivate our discussion with Hervé's classical results on the two-dimensional bidisc, [20]. In Sections 3.2 and 3.3 we provide a detailed treatment of the case of a finite product of Hilbert balls, which can be infinite-dimensional. The main results are a generalisation of the Wolff Theorem and the Denjoy-Wolff Theorem, which have been published in [12].

In Chapter 4, we investigate holomorphic dynamics on irreducible rank-2 bounded symmetric domains, which may be infinite-dimensional. These domains are the Lie ball and the open unit ball of the JBW*-triple $L(\mathbb{C}^2, H)$ for a Hilbert space $H$ of dimension at least 2. Together with the product of two Hilbert balls, which fall into the study of finite products of Hilbert balls studied in Chapter 3, they constitute the class of all infinite-dimensional rank-

2 bounded symmetric domains. The Lie ball has been studied in [10] and we review the results briefly in Section 4.1. The new results for the case of the open unit ball $D$ of $L(\mathbb{C}^2, H)$, which are discussed in Sections 4.2, 4.3, 4.4 and 4.5, describe explicitly in Jordan theoretic terms a family of *horospheres,* which are invariant under the application of the holomorphic map in question, and use this result to discuss the asymptotic behaviour of the iterates and the limit functions of the map in terms of the boundary components of $D$.

# CHAPTER 1

## Denjoy-Wolff Theorem and Symmetric Domains

In this chapter, we discuss holomorphic dynamics in one dimension, focusing on the celebrated Denjoy-Wolff Theorem, and introduce bounded symmetric domains and the topology of locally uniform convergence, as a setting for the infinite-dimensional holomorphic dynamics studied in this thesis.

## 1.1 Holomorphic dynamics in one dimension

We begin our discussion with the classical Wolff Theorem [42] and Denjoy-Wolff Theorem [13, 41] for the dynamics of a fixed-point-free holomorphic self-map on the open unit disc in the complex plane.

First we shall fix some notation. Let $D(c,r) = \{z \in \mathbb{C} : |z-c| < r\}$ be the open disc in $\mathbb{C}$ centred at $c$ of radius $r > 0$ and let $\partial D(c,r) = \{z \in \mathbb{C} : |z-c| = r\}$ be its boundary. The norm closure of $D(c,r)$ is denoted $\overline{D}(c,r)$ which is the closed disc $\{z \in \mathbb{C} : |z-c| \leq r\}$. For short we denote the open unit disc by $\mathbb{D} = D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$, the unit circle by $\partial\mathbb{D} = \partial D(0,1) = \{z \in \mathbb{C} : |z| = 1\}$ and the closed disc by $\overline{\mathbb{D}} = \overline{D}(0,1)$. Let $I : \mathbb{D} \to \mathbb{D}$ denote the identity map.

For a holomorphic map $f : \mathbb{D} \to \mathbb{D}$, we denote the $n$-th iterate of $f$ by

$$f^n = \overbrace{f \circ \cdots \circ f}^{n\text{-times}} \quad (n \in \mathbb{N}).$$

A holomorphic map $f : \mathbb{D} \to \mathbb{D}$ which is bijective is called an *automorphism* of $\mathbb{D}$. An important example of an automorphism is the *Möbius transformation* $g_a : \mathbb{D} \to \mathbb{D}$ induced by an element $a \in \mathbb{D}$ which is defined by

$$g_a(z) = \frac{z + a}{1 + \overline{a}z}.$$

Note that $g_0$ is the identity map. We can naturally extend the domain of $g_a$ to $\overline{\mathbb{D}}$ without difficulty and then $g_a(\partial\mathbb{D}) = \partial\mathbb{D}$ injectively. The inverse of $g_a$ is $g_{-a}$. It is well known that all automorphisms of the unit disc are of the form $\alpha g_a$ where $\alpha \in \partial\mathbb{D}$, see for example [37, Theorem 12.6]. Moreover, it can be proven that an automorphism without fixed point in $\mathbb{D}$, when extended to $\overline{\mathbb{D}}$, has at least one and at most two distinct fixed points in the boundary $\partial\mathbb{D}$. In fact, as a consequence of a more general result we shall prove later, *q.v.* Lemma 3.3.12, an automorphism given by $\alpha g_a$ with $a \in \mathbb{D}\backslash\{0\}$ is fixed-point-free if and only if $|1 - \alpha| \leq 2|a|$. In particular, $\alpha g_a$ has only one fixed point in $\partial\mathbb{D}$ if $|1 - \alpha| = 2|a|$ but has two distinct fixed points in $\partial\mathbb{D}$ if $|1 - \alpha| < 2|a|$.

Let $\rho$ be the *Poincaré distance* on $\mathbb{D}$ defined by

$$\rho(z, w) = \tanh^{-1} |g_{-z}(w)| \quad (z, w \in \mathbb{D}).$$

The open unit disc $\mathbb{D}$ equipped with the Poincaré distance gives us the simplest model for hyperbolic geometry.

By the Schwarz-Pick Lemma stated below, *q.v.* [38, Theorem 8.1.4], each holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is nonexpansive in the Poincaré distance $\rho$.

**Lemma** (Schwarz-Pick Lemma)**.** *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then*

$$\rho(f(z), f(w)) \leq \rho(z, w)$$

*for all $z, w \in \mathbb{D}$.*

In [42] Wolff showed the existence of a class of invariant domains for a fixed-point-free, holomorphic function, $f : \mathbb{D} \to \mathbb{D}$. Before stating the theorem, we now sketch the main idea from our perspective of hyperbolic geometry with a view to later infinite-dimensional extension.

Choose a sequence $(a_k)$ in $(0, 1)$ strictly increasing to 1 and let $f_k = a_k f$. The calculation

$$|I - (I - f_k)| = |f_k| = a_k|f| < a_k = |I| \quad \text{on } \partial D(0, a_k)$$

and an application of Rouché's Theorem gives that $(I - f_k)$ and $I$ have the same number of zeros in $D(0, a_k)$. Therefore $f_k$ has exactly one fixed point in $D(0, a_k)$. Call this fixed point $z_k$ and, passing to a subsequence of $(z_k)$, if necessary, we may assume without loss of generality that $\lim_k z_k = \xi$ for some $\xi \in \overline{\mathbb{D}}$. In fact $\xi \in \partial\mathbb{D}$ because if $\xi \in \mathbb{D}$ then $f(\xi) = \lim_k a_k f(z_k) = \lim_k f_k(z_k) = \lim_k z_k = \xi$ which is not permitted as $f$ is fixed-point-free.

For each $k \in \mathbb{N}$ and $y \in \mathbb{D}$, the closed *Poincaré disc* with centre $z_k$ and radius $\rho(y, z_k)$ is defined by

$$D_k(y) := \{w \in \mathbb{C} \;:\; \rho(w, z_k) \leq \rho(y, z_k)\}$$

and a simple calculation shows that

$$D_k(y) = g_{z_k}\left(\overline{D}(0, |g_{-z_k}(y)|)\right).$$

These *Poincaré discs* are actually closed Euclidean discs, with adjusted centre and radius. The Schwarz-Pick Lemma gives

$$\rho(f_k^n(y), z_k) = \rho(f_k^n(y), f_k^n(z_k)) \quad \leq \quad \rho(y, z_k) \quad (y \in \mathbb{D},\, k, n \in \mathbb{N}), \tag{1.1.1}$$

which confirms that $f_k^n(y) \in D_k(y)$ and it is obvious that $y$ is on the boundary of $D_k(y)$ for all $k \in \mathbb{N}$. The important observation is that $S(\xi, y)$, the *horocycle* at $\xi$, can be well-defined as the limit of a sequence of *Poincaré discs* in the following sense:

$$S(\xi, y) \quad = \quad \{x \in \overline{\mathbb{D}} \,:\, x = \lim_k x_k \text{ and } x_k \in D_k(y)\}.$$

The horocycle $S(\xi, y)$ turns out to be the closed Euclidean disc in $\overline{\mathbb{D}}$ with centre $t^2\xi$ and radius $1 - t^2$ where $t^2 = \frac{1-|y|^2}{1-|y|^2+|\xi-y|^2}$. Indeed, using the readily verifiable formula (*cf.* (3.2.1))

$$g_z(rx) = \frac{1-r^2}{1-r^2|z|^2} z + r\, \frac{1-|z|^2}{1-r^2|z|^2}\, g_{rz}(x) \quad (0 < r < 1, x \in \overline{\mathbb{D}}),$$

we have

$$\begin{aligned} D_k(y) \quad &= \quad g_{z_k}\left(\overline{D}(0, r_k)\right) \\ &= \quad \overline{D}\left(t_k^2 z_k,\, (1-t_k^2) r_k^{-1}\right) \end{aligned}$$

where $r_k = |g_{-z_k}(y)|$ and $t_k^2 = \frac{1-r_k^2}{1-r_k^2|z_k|^2}$. The aforementioned description of the horocycle then follows once we acknowledge the limits

$$\begin{aligned} r_k \quad &= \quad |g_{-z_k}(y)| = \left| z_k\, \frac{y - z_k}{|z_k|^2 y - z_k} \right| \to |\xi| = 1, \\ t_k^2 \quad &= \quad \frac{1-r_k^2}{1-r_k^2|z_k|^2} = \frac{1-|y|^2}{1-|y|^2+|1-y\overline{z_k}|^2} \to \frac{1-|y|^2}{1-|y|^2+|\xi-y|^2}. \end{aligned}$$

An immediate consequence is that $S(\xi, y)$ is internally tangent to the unit circle $\partial\mathbb{D}$ at $\xi$. Moreover, using the fact that

$$1 - |y|^2 + |\xi - y|^2 \;=\; 2(1 - \operatorname{Re} y\overline{\xi})$$

and

$$(1 - t^2)(1 - |y|^2 + |\xi - y|^2) = |\xi - y|^2,$$

the calculation

$$\begin{aligned} |y - t^2\xi|^2 &= |y - \xi + (1 - t^2)\xi|^2 \\ &= (1 - t^2)^2 + |y - \xi|^2 - 2(1 - t^2)\mathrm{Re}(1 - y\overline{\xi}) \\ &= (1 - t^2)^2 + |y - \xi|^2 - (1 - t^2)(1 - |y|^2 + |\xi - y|^2) \\ &= (1 - t^2)^2 \end{aligned}$$

shows that $y$ is on the boundary of $S(\xi, y)$. In fact, every other $y' \in \mathbb{D}$ on the boundary of the horocycle $\partial S(\xi, y)$ gives rise to the same horocycle, that is $S(\xi, y) = S(\xi, y')$, as there is only one circle that is internally tangent to $\partial\mathbb{D}$ at $\xi$ and has $y'$ on the boundary.

We will now show that the horocycle $S(\xi, y)$ is $f$-invariant in the sense that $f(S(\xi, y) \cap \mathbb{D}) \subset S(\xi, y) \cap \mathbb{D}$. Suppose $x \in S(\xi, y) \cap \mathbb{D}$, then there exists $x_k \in D_k(y)$ such that $x = \lim_k x_k$. By the Schwarz-Pick Lemma and as $f_k(z_k) = z_k$, we have

$$\rho(f_k(x_k), z_k) = \rho(f_k(x_k), f_k(z_k)) \leq \rho(x_k, z_k) \leq \rho(y, z_k),$$

which shows that $f_k(x_k) \in D_k(y)$. Taking limits gives $f(x) = \lim_k f(x_k) = \lim_k a_k f(x_k) = \lim_k f_k(x_k) \in S(\xi, y)$.

We summarise Wolff's beautiful result in the following theorem.

**Theorem 1.1.1** (Wolff's Theorem)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a fixed-point-free, holomorphic function. Then there exists a boundary point $\xi \in \partial\mathbb{D}$ such that every closed disc, internally tangent to $\partial\mathbb{D}$ at $\xi$ is $f$-invariant.*

**Example 1.1.2.** The Möbius transformation $g_a : \mathbb{D} \to \mathbb{D}$ is holomorphic and, when $a \neq 0$, is fixed-point-free. Set $a = \frac{1}{4}(1+i)$. In Figure 1.1.1 we show the $g_a$-invariant horocycles at the points $-1/2$ , $g_a(-1/2)$ and $g_a^2(-1/2)$.

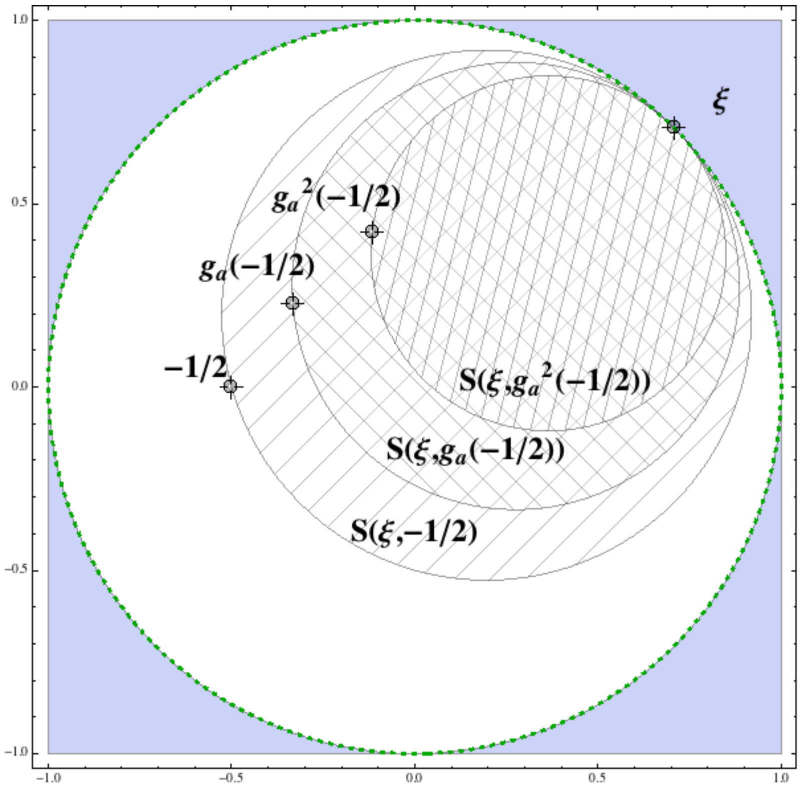



Figure 1.1.1: The $g_a$-invariant horocycles $S(\xi, y)$ at the points $y = -1/2$, $g_a(-1/2)$, $g_a^2(-1/2)$, where $a = \frac{1}{4}(1+i)$.

We are now in a position to state the Denjoy-Wolff Theorem and discuss its relationship to Theorem 1.1.1.

**Theorem 1.1.3** (Denjoy-Wolff Theorem)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a fixed-point-free holomorphic function. Then $(f^n)$ converges to a boundary point $\xi \in \partial\mathbb{D}$ uniformly on compact sets in $\mathbb{D}$.*

In the case where $f$ is an automorphism, the result follows from elementary calculations using the boundary fixed points, where it can be shown that one of the boundary fixed points is the limit $\xi$. In the case where $f$ is not an automorphism, let $h$ be a subsequential limit of $(f^n)$, that is there exists a subsequence $(f^{n_j})$ of $(f^n)$ converging to $h$ uniformly on compact sets in $\mathbb{D}$. It can be proven that every subsequential limit of $(f^n)$ is constant. Therefore there exists a $\zeta \in \overline{\mathbb{D}}$ such that $h(z) = \zeta$ for all $z \in \mathbb{D}$. If $\zeta \in \mathbb{D}$, then by the continuity of $f$ we have

$$f(\zeta) = f(h(z)) = f(\lim_j f^{n_j}(z)) = \lim_j f(f^{n_j}(z)) = \lim_j f^{n_j}(f(z)) = h(f(z)) = \zeta,$$

which is impossible, as $f$ is fixed-point-free. So we must have $\zeta \in \partial\mathbb{D}$. If $\zeta \neq \xi$, then there exists a closed disc $B \subsetneq \overline{\mathbb{D}}$ internally tangent to $\partial\mathbb{D}$ at $\xi$ such that $\zeta \notin B$ which gives the contradiction $\zeta = h(z) = \lim_{j\to\infty} f^{n_j}(z) \in B$, since $f^n(z) \in B \cap \mathbb{D}$ for all $z \in B \cap \mathbb{D}$ and $n \in \mathbb{N}$. Therefore we must have $\zeta = \xi$. We have just shown that every convergent subsequence of $(f^n)$, in the topology of uniform convergence on compact sets, must converge to $\xi$ and so the whole sequence $(f^n)$ must converge to $\xi$ uniformly on compact sets.

*Remark* 1.1.4. In fact the convergence of a single orbit $(f^n(a))$ for some $a \in \mathbb{D}$ is both necessary and sufficient for the sequence of iterates $(f^n)$ to converge uniformly on compact sets to a constant map taking value in the boundary $\partial\mathbb{D}$. We shall see later that this result holds for the more general case of a fixed-point-free holomorphic self-map on a Hilbert ball.

We now give some examples of the Denjoy-Wolff Theorem in action for different types of fixed-point-free holomorphic map. See Figure 1.1.2 for the first four iterates of these functions on a ball $B$ strictly contained in the interior of $\mathbb{D}$, that is $B \subset \mathbb{D}$ and the distance from $B$ to $\partial\mathbb{D}$ is positive.

**Example 1.1.5.** The Möbius transformation $g_a : \mathbb{D} \to \mathbb{D}$ with $a \neq 0$ is a fixed-point-free automorphism with two distinct fixed points $\pm a/|a| \in \partial\mathbb{D}$ when extended to the boundary. Considering the convergence of the orbit $(g_a^n(0))$ to $a/|a|$, we can conclude that the sequence of iterates $(g_a^n)$ converges uniformly on compact sets to $a/|a|$. In particular, the Möbius transformation $g_a : \mathbb{D} \to \mathbb{D}$ with $a = \frac{1}{4}(1+i) = \frac{1}{2\sqrt{2}}e^{i\pi/4}$, given in Example 1.1.2, converges uniformly on compact sets to $e^{i\pi/4}$. See Figure 1.1.2(a).

**Example 1.1.6.** The automorphism given by $\alpha g_a : \mathbb{D} \to \mathbb{D}$ where $\alpha = \frac{1}{2}(1 + i\sqrt{3})$ and $a = 1/2$ has no fixed points in $\mathbb{D}$ but does have a unique fixed point $p \in \partial\mathbb{D}$ when extended to the boundary. Therefore the sequence of iterates $((\alpha g_a)^n)$ converges uniformly on compact sets to the boundary fixed point $p = \frac{\alpha - 1}{2\bar{a}} = \frac{1}{2}(-1 + i\sqrt{3})$. See Figure 1.1.2(b).

**Example 1.1.7.** The holomorphic function $f : \mathbb{D} \to \mathbb{D}$ defined by $f(z) = \frac{1}{2}(z+1)$ is fixed-point-free but not an automorphism. The image $f(\mathbb{D})$ is the open disc $D(\frac{1}{2}, \frac{1}{2})$ the norm closure of which touches the boundary only at 1, which necessitates 1 being the boundary point $\xi$ in Theorem 1.1.1. Therefore $(f^n)$ converges uniformly on compact sets to 1. See Figure 1.1.2(c).

This can, of course, be proven directly once one observes

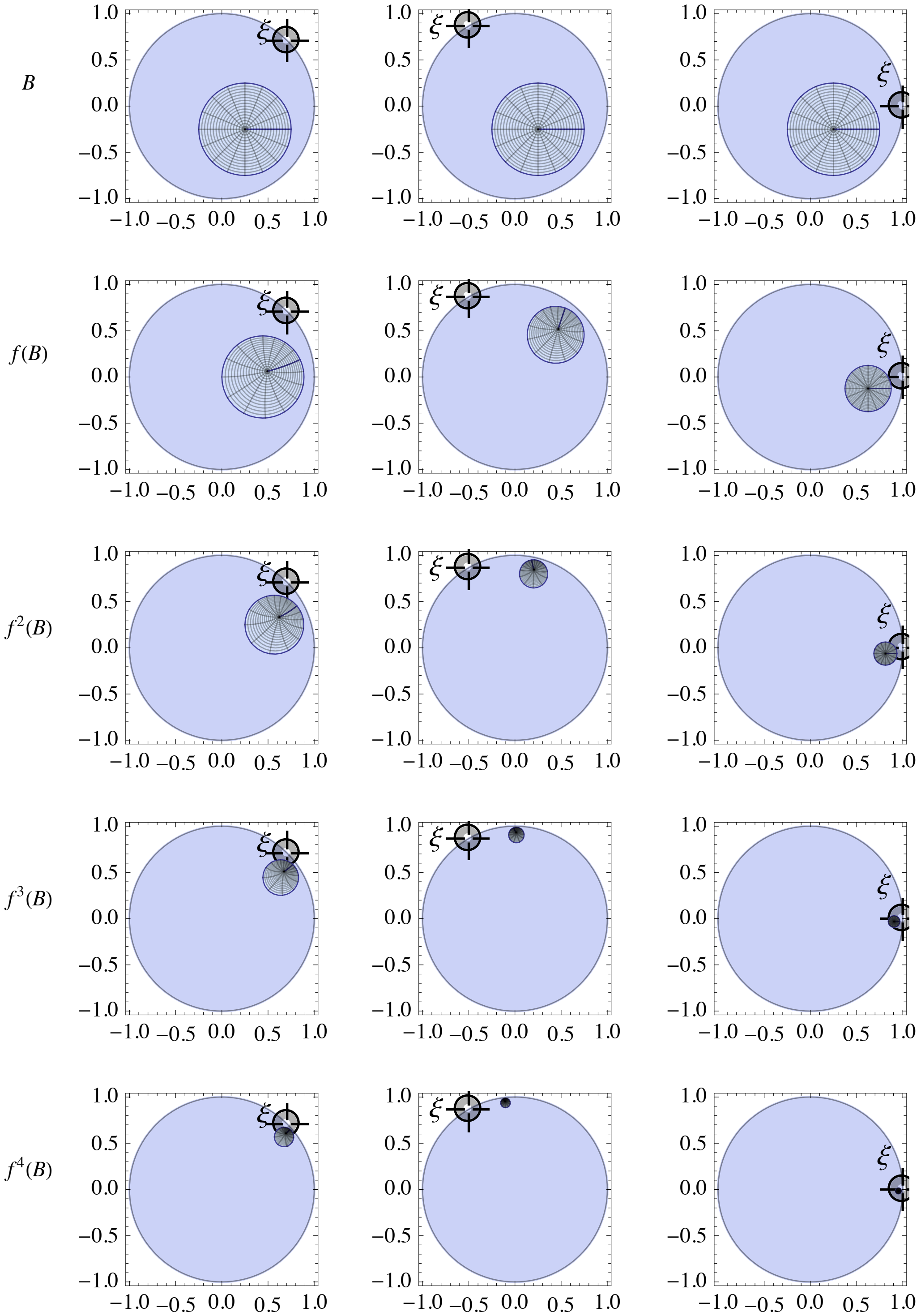
$$f^n(z) = \frac{1}{2^n}z + \sum_{r=1}^{n} \frac{1}{2^r} \quad (z \in \mathbb{D}, n \in \mathbb{N}).$$

However the former approach puts the hyperbolic geometry of $\mathbb{D}$ on centre stage.

Figure 1.1.2: Fixed-point-free holomorphic iteration in $\mathbb{D}$.

First four iterates of the ball $B$ under the holomorphic map $f : \mathbb{D} \to \mathbb{D}$ where

| (a) $f = g_a$ with $a = \frac{1}{4}(1+i)$. | (b) $f = \alpha g_a$ with $\alpha = \frac{1}{2}(1+i\sqrt{3})$ and $a = \frac{1}{2}$. | (c) $f$ is defined by $f(z) = \frac{1}{2}(z+1)$. |
|---|---|---|



Although we shall mainly be concerned with the fixed-point-free case, it would be remiss not to mention what is known about the iteration of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with a fixed point $z_0 \in \mathbb{D}$. It can be shown that the identity function $I : \mathbb{D} \to \mathbb{D}$ is the only holomorphic function with more than one fixed point. Excluding the case where $f = I$, we then have that $z_0$ is the unique fixed point of $f$. If $f$ is not an automorphism, then $(f^n)$ converges uniformly on compact sets to $z_0$. If $f$ is an automorphism, then either $f$ has period-$n$, that is $f^n = I$ for some $n \in \mathbb{N}$, or the set of iterates $\{f^n : n \in \mathbb{N}\}$ is dense in the compact group of all automorphisms of $\mathbb{D}$ which fix $z_0$. See [5] for further details.

We now present a new result on exactly which automorphisms have period-$n$. We prove a lemma first.

**Lemma 1.1.8.** *The $n$-th iterate of $\alpha g_a$ takes the form*

$$(\alpha g_a)^n(z) \quad = \quad \frac{\alpha E_n a + B_n z}{D_n + E_n z\overline{a}} \quad (n \geq 1)$$

*where $B_n$, $D_n$ and $E_n$ are independent of $z$ and are given by the following recursion relationships*

$$\begin{aligned} B_0 := 1, \quad B_1 = \alpha, \quad B_n &= \alpha\left(B_{n-1} + |a|^2 \sum_{r=0}^{n-2} B_r\right) && (n \geq 2) \\ D_1 = 1, \quad D_n &= 1 + \alpha|a|^2 \sum_{r=1}^{n-1}\sum_{s=0}^{r-1} B_s && (n \geq 2) \\ E_n &= \sum_{r=0}^{n-1} B_r. && (n \geq 1) \end{aligned}$$

*Proof.* We prove this lemma by induction on $n$. By definition, the first iterate of $\alpha g_a$ takes the form

$$\alpha g_a(z) \quad = \quad \frac{\alpha a + \alpha z}{1 + z\overline{a}}$$

which gives $B_1 = \alpha$, $D_1 = 1$ and $E_1 = 1 = B_0$, as required.

Suppose there exists some $n \in \mathbb{N}$ such that the result holds for all $k \leq n$, then

$$\begin{aligned}(\alpha g_a)^{n+1}(z) &= \alpha g_a\left(\frac{\alpha E_n a + B_n z}{D_n + E_n z\overline{a}}\right) \\ &= \frac{\alpha(D_n + \alpha E_n)a + \alpha(B_n + E_n|a|^2)z}{D_n + \alpha E_n|a|^2 + (B_n + E_n)z\overline{a}}.\end{aligned}$$

If we can prove the veracity of the equations

$$\begin{aligned}B_{n+1} &= \alpha\left(B_n + E_n|a|^2\right) \\ D_{n+1} &= D_n + \alpha E_n|a|^2 \\ E_{n+1} &= D_n + \alpha E_n = B_n + E_n,\end{aligned}$$

then the result holds for $(n+1)$-th case and we are done. Indeed the following calculations complete the proof.

$$\begin{aligned}\alpha\left(B_n + E_n|a|^2\right) &= \alpha\left(B_n + |a|^2\sum_{r=0}^{n-1} B_r\right) \\ &= B_{n+1},\end{aligned}$$

$$\begin{aligned}D_n + \alpha E_n|a|^2 &= 1 + \alpha|a|^2\sum_{r=1}^{n-1}\sum_{s=0}^{r-1} B_s + \alpha|a|^2\sum_{r=0}^{n-1} B_r \\ &= 1 + \alpha|a|^2\sum_{r=1}^{n}\sum_{s=0}^{r-1} B_s \\ &= D_{n+1},\end{aligned}$$

$$\begin{aligned}D_n + \alpha E_n &= 1 + \alpha|a|^2\sum_{r=1}^{n-1}\sum_{s=0}^{r-1} B_s + \alpha\sum_{r=0}^{n-1} B_r \\ &= 1 + B_1 + \sum_{r=1}^{n-1}\alpha(|a|^2\sum_{s=0}^{r-1} B_s + B_r) \\ &= B_0 + B_1 + \sum_{r=1}^{n-1} B_{r+1} \\ &= \sum_{r=0}^{n} B_r \\ &= E_{n+1},\end{aligned}$$

$$\begin{aligned}
B_n + E_n &= B_n + \sum_{r=0}^{n-1} B_r \\
&= \sum_{r=0}^{n} B_r \\
&= E_{n+1}.
\end{aligned}$$

□

**Corollary 1.1.9.** *The $E_n$ from Lemma 1.1.8 satisfy the following recursion relation*

$$E_0 := 0, E_1 = 1, E_2 = 1 + \alpha \qquad E_n = E_{n-1} + |a|^2 \sum_{r=1}^{n-2} E_{n-1-r}\alpha^r + \alpha^{n-1} \quad (n \geq 3).$$

*Proof.* From the proof of Lemma 1.1.8, and using the notation there, the $E_n$ follow the recursion relationship

$$\begin{aligned}
E_n &= E_{n-1} + B_{n-1} && (n \geq 1) \\
B_n &= \alpha\left(B_{n-1} + E_{n-1}|a|^2\right) && (n \geq 1).
\end{aligned}$$

Therefore

$$\begin{aligned}
E_n &= E_{n-1} + B_{n-1} \\
&= E_{n-1} + \alpha\left(B_{n-2} + E_{n-2}|a|^2\right) \\
&= E_{n-1} + \alpha E_{n-2}|a|^2 + \alpha B_{n-2} \\
&= E_{n-1} + \alpha E_{n-2}|a|^2 + \alpha(\alpha\left(B_{n-3} + E_{n-3}|a|^2\right)) \\
&= E_{n-1} + \alpha E_{n-2}|a|^2 + \alpha^2 E_{n-3}|a|^2 + \alpha^2 B_{n-3} \\
&= E_{n-1} + |a|^2 \sum_{r=1}^{n-2} E_{n-1-r}\alpha^r + \alpha^{n-2} B_1 \\
&= E_{n-1} + |a|^2 \sum_{r=1}^{n-2} E_{n-1-r}\alpha^r + \alpha^{n-1}.
\end{aligned}$$

□

**Proposition 1.1.10.** *Let $\alpha g_a : \mathbb{D} \to \mathbb{D}$ be an automorphism with a fixed point, where $\alpha \in \partial\mathbb{D}$ and $a \in \mathbb{D}\backslash\{0\}$. Then $\alpha g_a$ has period-$n$ if and only if $E_n = 0$ where $E_n$ is given in Lemma 1.1.8.*

*Proof.* If $(\alpha g_a)^n = I$, then

$$\frac{\alpha E_n a + B_n z}{D_n + E_n z\overline{a}} = z.$$

In particular this must hold for $z = 0$, giving $E_n = 0$.

Conversely, if $E_n = 0$ then

$$(\alpha g_a)^n z = \frac{B_n z}{D_n}$$

for all $z \in D$. As there exists a nonzero fixed point $z_0$ of $\alpha g_a$, we have $z_0 = \frac{B_n}{D_n} z_0$ which implies $B_n = D_n$ and $(\alpha g_a)^n = I$.

□

It is of interest to give the following explicit formula for $E_n$ in terms of $\alpha$ and $a$.

**Proposition 1.1.11.** *The $E_n$ from Lemma 1.1.8 satisfy*

$$E_n = \sum_{r=0}^{n-1} \alpha^r \sum_{s=0}^{\min\{r,n-1-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} \quad (n \geq 1). \tag{1.1.2}$$

The proof is straightforward but lengthy and is therefore included in the Appendix.

*Remark* 1.1.12. The automorphism $\alpha I$ has period-$n$ if and only if $\alpha$ is an $n$-th root of unity. Provided we exclude $\alpha = 1$ and permit $a = 0$ in (1.1.2) this is equivalent to $E_n = 1 + \alpha + \cdots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} = 0$.

By Proposition 1.1.10, Proposition 1.1.11 and Remark 1.1.12 we now have the following criterion for $n$-periodicity.

**Theorem 1.1.13.** *Let $\alpha g_a : \mathbb{D} \to \mathbb{D}$ be an automorphism with a fixed point that is not the identity, where $\alpha \in \partial\mathbb{D}$ and $a \in \mathbb{D}$. Then $\alpha g_a$ has period-$n$ if and only if*

$$\sum_{r=0}^{n-1} \alpha^r \sum_{s=0}^{\min\{r,n-1-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} = 0. \tag{1.1.3}$$

It may be of interest to note that the left hand side of (1.1.3) gives us the Pascal-like triangle:

| **n** | $\mathbf{E_n}$ |
|---|---|
| 1 | $1$ |
| 2 | $\alpha + 1$ |
| 3 | $\alpha^2 + \alpha\left(\lvert a\rvert^2 + 1\right) + 1$ |
| 4 | $\alpha^3 + \alpha^2\left(2\lvert a\rvert^2 + 1\right) + \alpha\left(2\lvert a\rvert^2 + 1\right) + 1$ |
| 5 | $\alpha^4 + \alpha^3\left(3\lvert a\rvert^2 + 1\right) + \alpha^2\left(\lvert a\rvert^4 + 4\lvert a\rvert^2 + 1\right) + \alpha\left(3\lvert a\rvert^2 + 1\right) + 1$ |
| $\vdots$ | $\vdots$ |

Pascal's triangle is visible if we were to allow $|a| = 1$, as then the coefficients of the powers of $\alpha$ in (1.1.3) are the binomial coefficients because

$$\sum_{s=0}^{\min\{r,n-1-r\}} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} = \begin{pmatrix} n-1 \\ r \end{pmatrix}$$

by basic properties of $\binom{n}{k}$ and Vandermonde's Convolution, which states that

$$\sum_{s=0}^{k} \begin{pmatrix} n \\ s \end{pmatrix} \begin{pmatrix} m \\ k-s \end{pmatrix} = \begin{pmatrix} n+m \\ k \end{pmatrix}$$

for non-negative integers $n, m$ and $k$.

*Remark* 1.1.14. From (1.1.3) we see that if $\alpha g_a$ is a period-$n$ automorphism, then so is $\beta g_b$ where $\beta \in \{\alpha, \overline{\alpha}\}$ and $b \in \{e^{i\theta}a \ : \ 0 < \theta \leq 2\pi\}$. Indeed, if $\alpha g_a$ has a fixed point we have $|1-\alpha| > 2|a|$ and, as $|1-\alpha| = |1-\overline{\alpha}|$ and $|a| = |e^{i\theta}a|$, the stated $\beta g_b$ also has a fixed point.

**Example 1.1.15.** Excluding the identity function, the period-$n$ automorphisms consist precisely of those automorphisms $\alpha g_a$ described by (1.1.3), or equivalently from the $n$-th line of the Pascal-like triangle above. Evidently all involutive automorphisms, except for the identity function, are of the form $-g_a$ with $a \in \mathbb{D}$, and, for example, all period-3 automorphisms, bar the identity function, can be expressed as $\alpha g_a$ with $a \in \mathbb{D}$ and $\alpha = \frac{1}{2}\left(-(1+|a|^2) \pm i\sqrt{4-(1+|a|^2)^2}\right)$.

## 1.2 Symmetric domains

A *domain* in a complex Banach space is a nonempty open connected set. Let $V$ and $W$ be complex Banach spaces and $U$ be an open subset of $V$. A map $f : U \to W$ is called *holomorphic* if $f$ is Fréchet differentiable in $U$, that is, if for each $a \in U$ there exists a continuous linear map $f'(a) : V \to W$ which satisfies

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - f'(a)(h)\|}{\|h\|} = 0.$$

If $f'(a)$ exists, then it is unique and is called the *derivative* of $f$ at $a$. A bijection $f : D \to D$ on a domain is called *biholomorphic* if both $f$ and its inverse are holomorphic. A *symmetric domain* $D$ in a complex Banach space

$E$ is a domain such that each point $a \in D$ is an isolated fixed point of an involutive biholomorphic map $s_a : D \to D$. If $D$ is a bounded symmetric domain, we call $s_a$ *the symmetry* at $a$, where the uniqueness is guaranteed by Cartan's uniqueness theorem, *q.v.* [17, Proposition III.2.1], which states that if a holomorphic map $f$ and biholomorphic map $g$, both from a bounded domain $\Omega$ to another domain in a complex Banach space, satisfy $f(p) = g(p)$ and $f'(p) = g'(p)$ for some $p \in \Omega$, then $f = g$. Indeed, every symmetry at $a$ fixes $a$ and has the same derivative at $a$, namely $-I$, where $I$ is the identity map on $E$.

**Example 1.2.1.** The open unit disc $\mathbb{D}$ of the complex plane is a bounded symmetric domain. The symmetry at 0 is simply

$$s_0 = -I$$

where $I$ is the identity map. The symmetry at a point $a \in \mathbb{D}$ is given by

$$s_a = g_a \circ s_0 \circ g_{-a}$$

where $g_a$ is the Möbius transformation at $a$ and $s_a$ can be written in the form $\alpha g_b$ with $\alpha = -1$ and $b = -g_a(a) = -\frac{2a}{1+|a|^2}$.

**Example 1.2.2.** All Euclidean balls and polydiscs, the latter being Cartesian products of the complex disc $\mathbb{D}$, are bounded symmetric domains.

Two domains $D$ and $D'$ are called *biholomorphic* if there is a biholomorphic map between them. A symmetric domain is called *irreducible* if it is not biholomorphic to a Cartesian product of symmetric domains. Élie Cartan classified in [6] the finite-dimensional irreducible bounded symmetric domains. The classification can be reformulated as the following theorem.

**Theorem 1.2.3.** *Let $D$ be a finite-dimensional irreducible bounded symmetric domain. Then $D$ is biholomorphic to the open unit ball of one of the following* *complex Banach spaces of matrices.*

- $M_{mn}(\mathbb{C})$, the space of $m \times n$ matrices with the operator norm for bounded linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$, where $1 \leq n < \infty$ and $n \leq m$,
- $S_n(\mathbb{C})$, the norm-closed subspace of $M_n(\mathbb{C})$ consisting of all $n \times n$ skew-symmetric matrices for $n \geq 2$,
- $H_n(\mathbb{C})$, the norm-closed subspace of $M_n(\mathbb{C})$ consisting of all $n \times n$ symmetric matrices,
- $Sp_n(H)$, the $n$-dimensional triple spin factor for $n \geq 3$,
- $M_{12}(\mathcal{O})$, the $1 \times 2$ matrices over the complex octonians, and
- $H_3(\mathcal{O})$, the $3 \times 3$ hermitian matrices over the complex octonians.

We shall see a generalisation of this result to bounded symmetric domains in reflexive Banach spaces of arbitrary dimension in Theorem 2.3.2, which employed Jordan theory. Indeed, we shall see that each bounded symmetric domain in a Banach space is biholomorphic to the open unit ball of a Banach space equipped with a suitable Jordan structure.

For later applications, we recall the definition of different types of boundary component of a domain. Let $V$ be a complex Banach space and $B \subset V$ a convex domain. Let $\mathcal{F}$ be a family of mappings from $\mathbb{D}$ to $V$ with image in $\overline{B}$.

**Definition 1.2.4.** Let $A$ be a subset of the norm closure $\overline{B}$ of $B$ satisfying

(i) $A \neq \emptyset$,

(ii) The image of every $f \in \mathcal{F}$ is contained in either $A$ or $\overline{B} \backslash A$, and

(iii) $A$ is minimal with respect to the properties in (i) and (ii).

We call $A$

- a *holomorphic boundary component of* $B$ if $\mathcal{F}$ is the set of all holomorphic mappings $f : \mathbb{D} \to V$ with $f(\mathbb{D}) \subset \overline{B}$,

- a *complex affine boundary component of* $B$ if $\mathcal{F}$ is the set of all complex affine mappings $f : \mathbb{D} \to \overline{B}$.

The disjoint union of the boundary components of $B$ equals $\overline{B}$. It turns out that, when $B$ is the open unit ball of $V$, its holomorphic and complex affine boundary components agree, *q.v.* [28, Proposition 4.2]. Henceforth, reference to "boundary component" shall always refer to this special case.

## 1.3 Topology of locally uniform convergence

Let $D$ be the open unit ball in a complex normed space $V$ and $C(D, V)$ be the complex vector space of all continuous maps of $D$ into $V$. The topological boundary of $D$ is denoted $\partial D$ and a nonempty subset $U \subset D$, whose distance to the boundary of $D$ is positive, that is $\inf\{\|u-v\| : u \in U \text{ and } v \in \partial D\} > 0$, is called *strictly contained* in $D$.

For any subset $K \subset D$, the map

$$f \in C(D, V) \mapsto \|f\|_K \in [0, \infty],$$

where

$$\|f\|_K = \sup\{\|f(x)\| : x \in K\},$$

is a semi-norm on the subspace of $C(D, V)$ on which $\|\cdot\|_K$ takes finite values.

Let $S = \{G_i : i \in I\}$ be a family of subsets $G_i \subset D$, which is closed with respect to finite unions. For $f_0 \in C(D, V)$, $G \in S$ and $\varepsilon > 0$ define

$$U(f_0, G, \varepsilon) := \{f \in C(D, V) : \|f - f_0\|_G < \varepsilon\}.$$

The family $\{U(f_0, G, \varepsilon) : G \in S \text{ and } \varepsilon > 0\}$ defines a fundamental system of neighbourhoods for a locally convex topology $\mathcal{T}$ of $C(D, V)$, which is not necessarily Hausdorff.

By choosing different families of subsets for $S$ we can define different topologies on $C(D, V)$.

- If $S$ is chosen as the family of all singleton subsets of $D$, then $\mathcal{T}$ is the *topology of pointwise convergence.*
- If $S$ is chosen as the family of all compact subsets of $D$, then $\mathcal{T}$ is the *topology of uniform convergence on compact subsets (compact-open topology).*
- If $S$ is chosen as the family of all finite unions of norm-closed balls strictly contained in $D$, then $\mathcal{T}$ is the *topology of locally uniform convergence.* This topology is finer than the compact-open topology in general, but the two are equivalent when $V$ is finite-dimensional.

The space of all holomorphic maps from $D$ into $V$, denoted by $Hol(D, V)$, is a closed subspace of $C(D, V)$ for both the compact-open topology and the topology of locally uniform convergence, [17, Proposition IV.3.1 and Lemma IV.3.3].

Let $f_n : D \to D \subset V$ be a sequence of holomorphic maps. Then $(f_n)$ converges to a holomorphic map $f : D \to \overline{D} \subset V$ in the topology of locally uniform convergence if and only if $(f_n)$ converges to $f$ uniformly on *any* ball strictly contained in $D$. We call a function $h : D \to \overline{D}$ a *limit function* of the above sequence $(f_n)$, if there is a subsequence $(f_{n_k})$ of $(f_n)$ converging to $h$ locally uniformly.

For a holomorphic map $f : D \to D$, we denote the $n$-th iterate of $f$ by

$$f^n = \overbrace{f \circ \cdots \circ f}^{n\text{-times}} \quad (n \in \mathbb{N}).$$

# CHAPTER 2

## Jordan Algebraic Structures

Since the introduction of Jordan algebras in quantum formalism in 1934 by P. Jordan, J. von Neumann and E. P. Wigner [22], unexpected applications have been found in many areas of mathematics. In particular, Jordan algebraic structures have been used to classify bounded symmetric domains in infinite-dimensional complex spaces and Jordan theory provides a useful tool to study holomorphic functions on these domains. In this chapter, we will discuss the relevant Jordan algebraic structures for our ensuing investigation of dynamics on some bounded symmetric domains. We begin by introducing the concept of a JB*-triple and discuss some examples in Sections 2.1 and 2.2. The Jordan algebraic classification of bounded symmetric domains will be explained in Section 2.3. *In this thesis, all vector spaces are over the field of complex numbers, unless otherwise stated.*

## 2.1 JB*-triples

JB*-triples are complex Banach spaces equipped with a Jordan algebraic structure. They play an important role in infinite-dimensional geometry and analysis. What follows is the essential Jordan theoretic tools that we rely on heavily in later chapters. We refer to [9] for a full treatment of the connection between

Jordan theory and Lie theory, and Jordan structures in geometry and analysis.

**Definition 2.1.1.** A complex vector space $V$ equipped with a triple product $\{\cdot,\cdot,\cdot\} : V^3 \longrightarrow V$ is called a *Hermitian Jordan triple* if the triple product is:

- linear and symmetric in the outer variables,
- conjugate linear in the middle variable, and
- satisfies the *main triple identity*:

$$\{a, b, \{x, y, z\}\} \quad = \quad \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

For subsets $A, B$ and $C$ of a Hermitian Jordan triple $V$ we define

$$\{A, B, C\} = \{\{a, b, c\} \ : \ a \in A, b \in B, \text{ and } c \in C\}.$$

A vector subspace $W$ of $V$ such that $x, y, z \in W$ implies $\{x, y, z\} \in W$ is called a *subtriple.* If $U$ is another Hermitian Jordan triple and $f : U \to V$ is a linear map which preserves the triple product:

$$f\{x, y, z\} = \{f(x), f(y), f(z)\}$$

for $x, y, z \in U$, then we call $f$ a *triple homomorphism.* A bijective triple homomorphism is called a *triple isomorphism.*

**Example 2.1.2.** The one-dimensional complex space $\mathbb{C}$ is a Hermitian Jordan triple in the product

$$\{x, y, z\} = x\overline{y}z \quad (x, y, z \in \mathbb{C}).$$

**Example 2.1.3.** A Jordan algebra $\mathcal{A}$ is a commutative algebra over a field $\mathbb{F}$ which satisfies the Jordan identity

$$(ab)a^2 = a(ba^2) \quad (a, b \in \mathcal{A}).$$

Hereinafter, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Any associative algebra $\mathcal{A}$ is a Jordan algebra in the *special Jordan product* $\circ$ defined by

$$a \circ b = \frac{1}{2}(ab + ba) \quad (a, b \in \mathcal{A}).$$

A Jordan algebra is called *special* if it is isomorphic to a Jordan subalgebra of some $(\mathcal{A}, \circ)$ and *exceptional* if it is not.

A complex Jordan algebra $\mathcal{A}$ equipped with an involution $^*$ is a Hermitian Jordan triple in the *canonical Hermitian Jordan triple product* defined by

$$\{a, b, c\} = (ab^*)c + a(b^*c) - b^*(ac) \quad (a, b, c \in \mathcal{A}). \tag{2.1.1}$$

We recall that an involution of $\mathcal{A}$ is a conjugate linear anti-automorphism of $\mathcal{A}$.

**Example 2.1.4.** Let $M_n(\mathbb{C})$ be the vector space of $n \times n$ complex matrices and $B^* = (\overline{b_{ji}})$ be the adjoint of a matrix $B = (b_{ij}) \in M_n(\mathbb{C})$. Then $M_n(\mathbb{C})$ is a Hermitian Jordan triple in the product

$$\{A, B, C\} = \frac{1}{2}\left(AB^*C + CB^*A\right). \tag{2.1.2}$$

The vector space $M_n(\mathbb{C})$ is actually a Jordan algebra with involution $^*$ when equipped with the special Jordan product $A \circ B = \frac{1}{2}(AB + BA)$. In this case, the triple product (2.1.2) agrees with the canonical Hermitian Jordan triple product given in (2.1.1). In fact, a C*-algebra $\mathcal{A}$ is a Hermitian Jordan triple in the product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$.

In Jordan theory, the following basic operators on a Hermitian Jordan triple $V$ play a fundamental role. Let $a, b \in V$.

- The *box operator* $a \,\Box\, b : V \longrightarrow V$ is defined by $(a \,\Box\, b)v = \{a, b, v\}$ for all $v \in V$.

- The *quadratic operator* induced by $a$ is denoted $Q_a : V \longrightarrow V$ and defined by $Q_a(v) = \{a, v, a\}$ for all $v \in V$.

- The *Bergmann operator* $B(a, b) : V \longrightarrow V$ is defined by

$$B(a, b)(v) = v - 2(a \,\square\, b)v + Q_a Q_b(v)$$

  for all $v \in V$. Note that $B(0, 0)$ is the identity operator.

In a Hermitian Jordan triple $V$, an element $e \in V$ is called a *tripotent* if $\{e, e, e\} = e$. We recall that a *matrix unit* in $M_n(\mathbb{C})$ is a matrix of the form $e_{k\ell} = (\delta_{ik}\delta_{j\ell})_{ij}$ for $k, \ell \in \{1, \ldots, n\}$, in other words, a matrix with one in the $k\ell$-th place and zero everywhere else. Matrix units in $M_n(\mathbb{C})$ are tripotents and represent the basic building blocks of $M_n(\mathbb{C})$. The tripotents in a C*-algebra coincide exactly with the partial isometries.

Given a tripotent $e$ in a Hermitian Jordan triple $V$, there correspond three important projections $P_k(e) : V \to V$ $(k = 0, 1, 2)$, called the *Peirce projections*, which are given by

$$\begin{aligned} P_2(e) &= Q_e^2 \\ P_1(e) &= 2(e \,\square\, e - Q_e^2) \\ P_0(e) &= I - 2(e \,\square\, e) + Q_e^2 = B(e, e) \end{aligned}$$

where $I : V \to V$ is the identity operator.

**Definition 2.1.5.** The *Peirce decomposition* of $V$ with respect to a tripotent $e$ is the following direct sum

$$V = V_0(e) \oplus V_1(e) \oplus V_2(e)$$

where $V_k(e) = P_k(e)V$ is called the *Peirce $k$-space* of $V$ with respect to $e$ for

$k = 0, 1, 2$. The Peirce $k$-spaces are the eigenspaces of the operator $2(e \,\square\, e)$:

$$V_k(e) = \{v \in V \;:\; 2(e \,\square\, e)v = kv\}.$$

The Peirce $k$-spaces obey the *Peirce multiplication rules*

$$\{V_i(e), V_j(e), V_k(e)\} \subset V_{i-j+k}(e), \quad \{V_0(e), V_2(e), V\} = \{V_2(e), V_0(e), V\} = \{0\},$$

for $i, j, k \in \{0, 1, 2\}$ and $V_i(e) = \{0\}$ for $i \notin \{0, 1, 2\}$.

**Definition 2.1.6.** A tripotent $e$ of a Hermitian Jordan triple $V$ is called

- *minimal* if $V_2(e)$ is one-dimensional,
- *maximal* (or *complete*) if $V_0(e) = \{0\}$, and
- *unitary* if $V_2(e) = V$.

The matrix units in $M_n(\mathbb{C})$ are all minimal tripotents. However, there are minimal tripotents which are not matrix units. For example, the matrix $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is also a minimal tripotent in $M_2(\mathbb{C})$.

**Definition 2.1.7.** Two elements $a$ and $b$ in a Hermitian Jordan triple $V$ are called *triple orthogonal to each other* if $a \,\square\, b = b \,\square\, a = 0$.

*Remark* 2.1.8. Two tripotents $u$ and $v$ in a Hermitian Jordan triple $V$ are triple orthogonal if and only if $\{u, v, v\} = 0$, *q.v.* [9, Corollary 1.2.46].

**Definition 2.1.9.** The *rank* of a Hermitian Jordan triple $V$ is defined as the maximal cardinality of a set of mutually triple orthogonal nonzero tripotents in $V$, which is unique.

*Remark* 2.1.10. If $u$ and $v$ are two triple orthogonal tripotents in a Hermitian Jordan triple $V$, then $\{u+v, u+v, u+v\} = \{u, u, u\} + \{v, v, v\} = u+v$, which implies that $u + v$ is also a tripotent.

For finite families of triple orthogonal tripotents we have the following useful joint Peirce decomposition, *q.v.* [32, Theorem 3.14].

**Theorem 2.1.11.** *Let $\{e_1, \ldots, e_n\}$ be a family of mutually triple orthogonal tripotents in a Hermitian Jordan triple $V$. Then we have*

$$V = \bigoplus_{0 \leq i \leq j \leq n} V_{ij}$$

*where the joint Peirce spaces $V_{ij}$ are defined by*

$$\begin{aligned}
V_{ii} =& V_2(e_i), && i = 1, \ldots, n;\\
V_{ij} =& V_{ji} = V_1(e_i) \cap V_1(e_j), && 1 \leq i < j \leq n;\\
V_{i0} =& V_{0i} = V_1(e_i) \cap \bigcap_{j \neq i} V_0(e_j), && i = 1, \ldots, n;\\
V_{00} =& V_0(e_1) \cap \cdots \cap V_0(e_n).
\end{aligned}$$

*The Peirce multiplication rules*

$$\{V_{ij}, V_{jk}, V_{k\ell}\} \subset V_{i\ell}$$

*hold and all other triple products which cannot be written in this form are zero. We denote by $P_{ij}$ the projection onto the space $V_{ij}$.*

*Let $M = \{0, 1, \ldots, n\}$ and $N \subset M \setminus \{0\}$. If $e_N = \sum_{i \in N} e_i$ then the Peirce $k$-spaces of $e_N$ are given by*

$$\begin{aligned}
V_2(e_N) &= \bigoplus_{i,j \in N} V_{ij},\\
V_1(e_N) &= \bigoplus_{\substack{i \in N\\ j \in M \setminus N}} V_{ij},\\
V_0(e_N) &= \bigoplus_{i,j \in M \setminus N} V_{ij}.
\end{aligned}$$

**Corollary 2.1.12.** *Let $\{e_1, \ldots, e_n\}$ be a family of mutually triple orthogonal tripotents in a Hermitian Jordan triple $V$ and let $x = \sum_{i=1}^n \lambda_i e_i$ where $\lambda_i \in \mathbb{C}$, and set $\lambda_0 = 0$. Then the Bergmann operator $B(x,x)$ satisfies*

$$B(x,x) \quad = \sum_{0 \le i \le j \le n} (1 - |\lambda_i|^2)(1 - |\lambda_j|^2) P_{ij}, \tag{2.1.3}$$

We now introduce the important concept of a JB*-triple.

**Definition 2.1.13.** A complex Banach space $V$ is called a *JB*-triple* if it is a Hermitian Jordan triple with a continuous triple product and the box operator $a \,\square\, a$ of each element $a \in V$ satisfies:

(i) $a \,\square\, a$ is a hermitian operator on $V$, that is, $\|e^{it(a \,\square\, a)}\| = 1$ for all $t \in \mathbb{R}$,

(ii) $a \,\square\, a$ has non-negative spectrum, and

(iii) $\|a \,\square\, a\| = \|a\|^2$.

In the Jordan approach to bounded symmetric domains, JB*-triples play a fundamental role due to the following Riemann Mapping Theorem of Kaup [25].

**Theorem 2.1.14.** *Let $D$ be a bounded domain in a complex Banach space. Then $D$ is a symmetric domain if and only if $D$ is biholomorphic to the open unit ball of a JB*-triple.*

**Example 2.1.15.** The previous examples of Hermitian Jordan triples, $\mathbb{C}$ and $M_n(\mathbb{C})$, are in fact JB*-triples. More generally, the Banach space $L(H)$ of all bounded linear operators on the Hilbert space $H$ is a JB*-triple in the triple product

$$\{A, B, C\} = \frac{1}{2}(AB^*C + CB^*A) \quad (A, B, C \in L(H)),$$

where $B^*$ is the adjoint of $B$. Moreover, every C*-algebra is a JB*-triple, as it is a closed subtriple of $L(H)$ for some Hilbert space $H$.

We note that the Peirce projections on a JB*-triple are contractive. Also, a linear bijection between two JB*-triples is an isometry if and only if it is a triple isomorphism [25, Proposition 5.5].

In a JB*-triple the Bergmann operator $B(x,x)$ is positive and the square root $B(x,x)^{1/2}$ exists. Indeed, keeping the same notation as in Corollary 2.1.12, we have the following formulae for the square roots

$$B(x,x)^{1/2} = \sum_{0\leq i\leq j\leq n} (1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2}P_{ij}, \tag{2.1.4}$$

$$B(x,x)^{-1/2} = \sum_{0\leq i\leq j\leq n} (1-|\lambda_i|^2)^{-1/2}(1-|\lambda_j|^2)^{-1/2}P_{ij}, \tag{2.1.5}$$

where the latter equation is valid if and only if $|\lambda_i| \neq 1$ for all $i \in \{1,\ldots,n\}$.

An important holomorphic mapping in complex analysis is the *Möbius transformation* of the open unit ball $D$ of a JB*-triple $V$. Let $a \in D$. The Möbius transformation $g_a : D \to D$ induced by $a$ is defined by

$$g_a(z) = a + B(a,a)^{1/2}(I + z \,\square\, a)^{-1}(z) \quad (z \in D), \tag{2.1.6}$$

where $I$ is the identity operator. We note that $g_0 = I$. The importance of the Möbius transformations can be seen from the fact that every automorphism of $D$ takes the form $\varphi \circ g_a$, where $a \in D$ and $\varphi$ is a linear isometry on $V$, *q.v.* [9, Proposition 3.2.6].

**Example 2.1.16.** The Möbius transformation of the open unit disc $\mathbb{D}$ induced by $a \in \mathbb{D}$ takes the familiar form

$$g_a(z) = \frac{z+a}{1+\bar{a}z} \quad (z \in \mathbb{D}),$$

which is consistent with our definition in Section 1.1.

By [28, Theorem 2.1], we have the following proposition.

**Proposition 2.1.17.** *Let $D$ be the open unit ball of a complex Banach space $E$. Then, for every boundary point $\zeta \in \partial D$, the locally uniform limit $g_\zeta := \lim_{a\to\zeta} g_a$ exists and additionally $g_\zeta : D \to E$ is a holomorphic mapping.*

It is important to note that the boundary components have the following useful description [28, Proposition 4.3].

**Proposition 2.1.18.** *Let $D$ be the open unit ball of a JB*-triple $V$. For every tripotent $e \in V$, the boundary component of $D$ containing $e$ is given by $K_e = e + D_0(e)$ where $D_0(e) = D \cap V_0(e)$ is the open unit ball of the Peirce 0-space of $e$. For every $a \in K_e$, we have $K_e = g_a(D) \subset \partial D$ if $e \neq 0$, and $K_0 = D$.*

We will make use of the spectral representation of elements in a JB*-triple, which have been shown in [24, 26].

**Theorem 2.1.19.** *For each element $x$ in a JB*-triple of finite rank $r$ there exist a family of mutually triple orthogonal minimal tripotents $\{e_i : i = 1, \ldots, r\}$ and a uniquely determined element $(t_1, \ldots, t_r) \in \mathbb{R}^r$ with $t_1 \geq t_2 \geq \cdots \geq t_r \geq 0$ such that*

$$x = \sum_{i=1}^{r} t_i e_i \tag{2.1.7}$$

*and*

$$\|x\| = t_1.$$

**Definition 2.1.20.** We refer to (2.1.7) as a *spectral decomposition* of $x$ and we call $t_1, \ldots, t_r$ the *spectral values* or the *triple spectrum* of $x$ in this decomposition.

Another important aspect of the Möbius transformation is that it can be used to describe the *Kobayashi distance* $\kappa$ on the open unit ball $D$ of a JB*-

triple, *cf.* [17, Chapter IV]:

$$\kappa(a,b) \quad = \quad \tanh^{-1}\|g_{-b}(a)\| \quad (a,b \in D).$$

We note that $\kappa$ is exactly the Poincaré distance on the complex unit disc $\mathbb{D}$.

**Definition 2.1.21.** A map $f : D \longrightarrow D$ is called *Kobayashi nonexpansive* if

$$\kappa(f(x), f(y)) \quad \leq \quad \kappa(x,y) \quad (x,y \in D)$$

and *Kobayashi contractive* if

$$\kappa(f(x), f(y)) \quad < \quad \kappa(x,y) \quad (\text{for } x \neq y).$$

We have the following Schwarz-Pick Lemma for JB*triples, *cf.* [9, Lemma 3.2.15].

**Lemma 2.1.22.** *All holomorphic self-maps on the open unit ball of a JB*-triple are Kobayashi nonexpansive.*

From [34, Proposition 3.1], we know that

$$1 - \|g_{-b}(a)\|^2 \quad = \quad \|B(a,a)^{-1/2}B(a,b)B(b,b)^{-1/2}\|^{-1} \quad (a,b \in D). \tag{2.1.8}$$

Moreover, we have, *q.v.* [9, Proposition 3.2.13],

$$\|B(a,a)^{-1/2}\| \quad = \quad \frac{1}{1 - \|a\|^2} \quad (a \in D). \tag{2.1.9}$$

**Definition 2.1.23.** A *JBW*-triple* is a JB*-triple with a Banach space predual, which is necessarily unique, *q.v.* [9, p.210].

We give several important examples of JB*-triples relevant to our study in the next section.

## 2.2 Examples of JB*-triples

Let $V$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and open unit ball $D$ with topological boundary $\partial D$. Then $V$ is a rank-1 JB*-triple in the triple product

$$\{x, y, z\} = \frac{1}{2} \left( \langle x, y \rangle z + \langle z, y \rangle x \right) \quad (x, y, z \in V).$$

The nonzero tripotents are exactly the boundary points of $D$. Let $e \in \partial D$. Then the Peirce projections are given by

$$\begin{aligned} P_2(e)x &= \langle x, e \rangle e \\ P_1(e)x &= x - \langle x, e \rangle e \\ P_0(e)x &= 0. \end{aligned}$$

Therefore, as $V_0(e) = \{0\}$ and $V_2(e) = \mathbb{C}e$, every nonzero tripotent $e$ is both maximal and minimal, and $V = V_1(e) \oplus \mathbb{C}e$, where $V_1(e)$ and $\mathbb{C}e$ are orthogonal with respect to the inner product.

We now show that $V$ is indeed a JB*-triple, which is also a JBW*-triple.

**Lemma 2.2.1.** *A complex Hilbert space $V$ with inner product $\langle \cdot, \cdot \rangle$ is a JB*-triple with triple product*

$$\{x, y, z\} = \frac{1}{2} \left( \langle x, y \rangle z + \langle z, y \rangle x \right) \quad (x, y, z \in V).$$

*Proof.* It is a straightforward exercise to confirm that $V$ is a Hermitian Jordan triple and the calculations $(a \,\square\, a)x = \frac{1}{2} \left( \|a\|^2 x + \langle x, a \rangle a \right)$ and

$$\langle (a \,\square\, a)x, x \rangle = \frac{1}{2} \left( \|a\|^2 \|x\|^2 + |\langle x, a \rangle|^2 \right) \in [0, \infty) \subset \mathbb{R}$$

show simultaneously that $a \,\square\, a$ has real numerical range, and is therefore a hermitian operator, *q.v.* [4], and has non-negative spectrum $\sigma(a \,\square\, a) \subset [m, M]$, where $m = \inf_{\|x\|=1}\langle (a \,\square\, a)x, x\rangle$ and $M = \sup_{\|x\|=1}\langle (a \,\square\, a)x, x\rangle$.

Finally $\|a \,\square\, a\| = \|a\|^2$ can be seen from the calculation

$$\begin{aligned}
\|a\|^2 &= \|(a \,\square\, a)\frac{a}{\|a\|}\| \\
&\le \|a \,\square\, a\| \\
&= \sup_{\|x\|\le 1} \|(a \,\square\, a)x\| \\
&= \sup_{\|x\|\le 1} \left\| \frac{1}{2}\left(\|a\|^2 x + \langle x, a\rangle a\right)\right\| \\
&\le \|a\|^2.
\end{aligned}$$

Of course we could have proved $\|e^{it(a\,\square\, a)}\| = 1$ for all $t \in \mathbb{R}$ directly. Indeed, this is obvious when $a = 0$, and when $a \neq 0$ we have

$$\begin{aligned}
(a \,\square\, a)^n x &= \frac{1}{2^n}\|a\|^{2n}x + \frac{2^n - 1}{2^n}\|a\|^{2(n-1)}\langle x, a\rangle a \\
&= \frac{1}{2^n}\|a\|^{2n}\left(x - \langle x, a\rangle \frac{a}{\|a\|^2}\right) + \|a\|^{2n}\langle x, a\rangle \frac{a}{\|a\|^2} \\
&= \frac{1}{2^n}\|a\|^{2n} P_1\left(\frac{a}{\|a\|}\right)x + \|a\|^{2n} P_2\left(\frac{a}{\|a\|}\right)x
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in V$. Therefore, for each $x \in V$,

$$\begin{aligned}
\exp\left[it(a \,\square\, a)\right] x &= \sum_{n=0}^{\infty} \frac{i^n t^n}{n!}(a \,\square\, a)^n(x) \\
&= P_1\left(\frac{a}{\|a\|}\right) x \sum_{n=0}^{\infty} \frac{i^n t^n}{n!}\frac{1}{2^n}\|a\|^{2n} + P_2\left(\frac{a}{\|a\|}\right) x \sum_{m=0}^{\infty} \frac{i^m t^m}{m!}\|a\|^{2m} \\
&= \exp\left[\frac{1}{2}it\|a\|^2\right] P_1\left(\frac{a}{\|a\|}\right)x + \exp\left[it\|a\|^2\right] P_2\left(\frac{a}{\|a\|}\right)x.
\end{aligned}$$

Hence, by orthogonality of $P_1\left(\frac{a}{\|a\|}\right)$ and $P_2\left(\frac{a}{\|a\|}\right)$, we have

$$\begin{aligned}
\|\exp\left[it(a \,\square\, a)\right] x\|^2 &= \left\|\exp\left[\frac{1}{2}it\|a\|^2\right] P_1\left(\frac{a}{\|a\|}\right)x + \exp\left[it\|a\|^2\right] P_2\left(\frac{a}{\|a\|}\right)x\right\|^2 \\
&= \left\|P_1\left(\frac{a}{\|a\|}\right)x\right\|^2 + \left\|P_2\left(\frac{a}{\|a\|}\right)x\right\|^2 \\
&= \|x\|^2.
\end{aligned}$$

□

In a Hilbert space $V$, the square root of the Bergmann operator $B(a,a)$ is given by

$$B(a,a)^{1/2}(z) = \sqrt{1-\|a\|^2}\left(z + (\sqrt{1-\|a\|^2}-1)\langle z,a\rangle \frac{a}{\|a\|^2}\right), \qquad (2.2.1)$$

where $a \neq 0$, *q.v.* [9, p.188]. In particular, we have $B(a,a)^{1/2}(a) = (1-\|a\|^2)a$ and

$$B(a,a)^{-1/2}(a) = \frac{a}{1-\|a\|^2}. \qquad (2.2.2)$$

The Möbius transformation $g_a$ induced by an element $a$ in the open unit ball $D$ of $V$ has the form

$$g_a(z) = a + \frac{B(a,a)^{1/2}}{1+\langle z,a\rangle} \quad (z \in D). \qquad (2.2.3)$$

In particular, the Möbius transformation $g_a$ induced by a nonzero element $a \in D$ has the form

$$g_a(z) = \frac{1}{1+\langle z,a\rangle}\left(a + P_2\left(\frac{a}{\|a\|}\right)(z) + \sqrt{1-\|a\|^2}P_1\left(\frac{a}{\|a\|}\right)(z)\right) \qquad (2.2.4)$$

for $z \in D$. In this case, the formula (2.1.8) for $1-\|g_{-b}(a)\|^2$ reduces to

$$1-\|g_{-b}(a)\|^2 = \frac{(1-\|a\|^2)(1-\|b\|^2)}{|1-\langle a,b\rangle|^2}. \qquad (2.2.5)$$

The next example of a JB*-triple is the $\ell_\infty$-sum of Hilbert spaces, which is relevant to our investigation later. Let $(V_j, \langle\cdot,\cdot\rangle_j)$ be a Hilbert space with open unit ball $D_j = \{z \in V_j : \|z\| < 1\}$ for $j \in \{1,\dots,p\}$. Let $D = D_1 \times \cdots \times D_p$ be a finite Cartesian product of Hilbert balls, which will be called a *polyball.* If each $D_j = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then $D$ is more commonly known as a *polydisc.* We observe that $D$ is the open unit ball of the $\ell_\infty$-sum of $(V_j)_{j=1}^p$ :

$$V := V_1 \oplus_\infty \cdots \oplus_\infty V_p,$$

where $\|v\| = \sup_{1\le j\le p}\|v_j\|$ for $v = (v_1,\dots,v_p) \in V$.

$V$ is a rank-$p$ JB*-triple and in fact a JBW*-triple, when equipped with the coordinatewise triple product

$$\{(x_1,\ldots,x_p),(y_1,\ldots,y_p),(z_1,\ldots,z_p)\} \quad = \quad (\{x_1,y_1,z_1\}_1,\ldots,\{x_p,y_p,z_p\}_p)$$

where $\{\cdot,\cdot,\cdot\}_j : V_j^3 \longrightarrow V_j$ is the Jordan triple product described above for a single Hilbert space, namely $\{x_j,y_j,z_j\}_j = \frac{1}{2}(\langle x_j,y_j\rangle z_j + \langle z_j,y_j\rangle x_j)$ for $x_j,y_j,z_j \in V_j$. We will suppress the subscript $j$ when the meaning is clear from the context.

We shall use boldface characters for elements in $V$, for instance $\boldsymbol{b} \in V$, and standard letters for elements in a single Hilbert space, for instance $b_1 \in V_1$. Given the coordinatewise nature of the Jordan triple product on $V$, it can be verified readily that the Bergmann operator $B(\boldsymbol{b},\boldsymbol{c})$ on $V$, the Möbius transformation $g_{\boldsymbol{a}}$ and the Kobayashi distance $\kappa(\boldsymbol{a},\boldsymbol{z})$ on $D$ are given coordinatewise by

$$\begin{aligned} B(\boldsymbol{b},\boldsymbol{c})\boldsymbol{x} &= (B(b_1,c_1)(x_1),\ldots,B(b_p,c_p)(x_p)) \\ g_{\boldsymbol{a}}(\boldsymbol{z}) &= (g_{a_1}(z_1),\ldots,g_{a_p}(z_p)) \\ \kappa(\boldsymbol{a},\boldsymbol{z}) &= \sup\{\kappa(a_1,z_1),\ldots,\kappa(a_p,z_p)\} \end{aligned}$$

for $\boldsymbol{a} = (a_1,\ldots,a_p)$, $\boldsymbol{b} = (b_1,\ldots,b_p)$, $\boldsymbol{c} = (c_1,\ldots,c_p)$, $\boldsymbol{x} = (x_1,\ldots,x_p)$ and $\boldsymbol{z} = (z_1,\ldots,z_p)$.

Next we give another example of a JB*-triple which is also relevant to our later investigation. Let $(H,\langle\cdot,\cdot\rangle_H)$ and $(K,\langle\cdot,\cdot\rangle_K)$ be Hilbert spaces and $L(H,K)$ be the Banach space of all bounded linear operators from $H$ to $K$. The space $L(H,K)$ becomes a JBW*-triple when equipped with the triple product

$$\{A,B,C\} = \frac{1}{2}(AB^*C + CB^*A) \quad (A,B,C \in L(H,K)),$$

where $B^* \in L(K,H)$ is the adjoint of $B$. The predual of $L(H,K)$ is the space $\mathcal{N}(K,H)$ of trace-class operators from $K$ to $H$. In fact, the following spaces are isometrically isomorphic

$$L(H,K) \cong (H\hat{\otimes}K^*)^* \cong (K^*\hat{\otimes}H)^* \cong \mathcal{N}(K,H)^*$$

where $\hat{\otimes}$ is the projective tensor product, *q.v.* [15, p.230; 40, p.179]. We shall discuss the predual of the space $L(\mathbb{C}^2, H)$ in greater detail in Section 4.2.

We now turn to the particular case of the rank-2 JBW*-triple $L(\mathbb{C}^2, H)$, namely the Banach space of all bounded linear operators from $\mathbb{C}^2$ to $H$. We will use $\langle\cdot,\cdot\rangle$ to denote the inner product in both Hilbert spaces when confusion is unlikely. For the remainder of this section $V := L(\mathbb{C}^2, H)$.

The tripotents in $V$ are elements $e$ such that $e = ee^*e$, where $ee^*$ is a projection on $H$ and $e^*e$ is a projection on $\mathbb{C}^2$. The Peirce projections of a single tripotent $e \in V$ are

$$P_2(e)(T) = ee^*Te^*e \tag{2.2.6}$$

$$P_1(e)(T) = (I_H - ee^*)Te^*e + ee^*T(I_{\mathbb{C}^2} - e^*e) \tag{2.2.7}$$

$$P_0(e)(T) = (I_H - ee^*)T(I_{\mathbb{C}^2} - e^*e) \tag{2.2.8}$$

where $I_H$ and $I_{\mathbb{C}^2}$ are the identity operators on $H$ and $\mathbb{C}^2$ respectively, *cf.* the matrix representation in [9, pp33-34].

We note that the rank-one operators in $V$ are of the form $a \otimes b : \mathbb{C}^2 \to H$ where $a \in \mathbb{C}^2\backslash\{0\}$ and $b \in H\backslash\{0\}$ and

$$(a \otimes b)(z) = \langle z, a\rangle b \quad (z \in \mathbb{C}^2).$$

One can easily verify that $\|a \otimes b\| = \|a\|\|b\|$ and the adjoint $(a \otimes b)^*$ is just $b \otimes a : H \to \mathbb{C}^2$ given by $(b \otimes a)(h) = \langle h, b\rangle a$ for $h \in H$. We note that the minimal tripotents in $V$ are exactly the rank-one operators $a \otimes b$ with

$\|a\|_{\mathbb{C}^2} = \|b\|_H = 1$ (*q.v.* [19, 24]). However, the representation of a minimal tripotent $e$ by $a_e \otimes b_e$ is unique only up to multiplication by a unimodular constant, in that $e$ can also be represented by $(wa_e) \otimes (wb_e)$ where $w \in \partial\mathbb{D}$. For convenience, we write $x \perp y$ to denote that $x$ and $y$ are orthogonal in a Hilbert space, not to be confused with triple orthogonality in a Hermitian Jordan triple.

**Lemma 2.2.2.** *Let $e_i = a_i \otimes b_i$ be a rank-one operator in $V$ for $i = 1, 2$. Then $e_1$ and $e_2$ are triple orthogonal if and only if $a_1 \perp a_2$ and $b_1 \perp b_2$ in their respective Hilbert spaces. In this case, we have $e_1(\mu) \perp e_2(\mu)$* in *$H$ for all $\mu \in \mathbb{C}^2$.*

*Proof.* Firstly, suppose $e_1$ and $e_2$ are triple orthogonal, then we have $0 = \{e_1, e_2, f\} = \frac{1}{2}(e_1 e_2^* f + f e_2^* e_1)$ for all $f \in V$. More explicitly we have

$$\begin{aligned} 0 &= \langle\langle f(\cdot), b_2\rangle a_2, a_1\rangle b_1 + f[\langle\langle \cdot, a_1\rangle b_1, b_2\rangle a_2] \\ &= \langle f(\cdot), b_2\rangle\langle a_2, a_1\rangle b_1 + \langle \cdot, a_1\rangle\langle b_1, b_2\rangle f(a_2). \end{aligned}$$

In particular when $f = e_1$ we have $0 = \langle \cdot, a_1\rangle\langle b_1, b_2\rangle\langle a_2, a_1\rangle b_1$, which clearly implies either $a_1 \perp a_2$ or $b_1 \perp b_2$. On the other hand if $f = e_2$ we have

$$\begin{aligned} 0 &= \langle\langle \cdot, a_2\rangle b_2, b_2\rangle\langle a_2, a_1\rangle b_1 + \langle \cdot, a_1\rangle\langle b_1, b_2\rangle\langle a_2, a_2\rangle b_2 \\ &= \langle \cdot, a_2\rangle\|b_2\|^2\langle a_2, a_1\rangle b_1 + \langle \cdot, a_1\rangle\langle b_1, b_2\rangle\|a_2\|^2 b_2. \end{aligned}$$

This gives us that $a_1 \perp a_2$ and $b_1 \perp b_2$. In this case we also have $\langle e_1(\mu), e_2(\mu)\rangle = \langle\langle \mu, a_1\rangle b_1, \langle \mu, a_2\rangle b_2\rangle = \langle \mu, a_1\rangle\overline{\langle \mu, a_2\rangle}\langle b_1, b_2\rangle = 0$, for $\mu \in \mathbb{C}^2$.

Secondly, suppose $a_1 \perp a_2$ *and* $b_1 \perp b_2$. Given any $x \in V$, we have

$$\begin{aligned} 2(e_1 \square e_2)(x) &= 2\{e_1, e_2, x\} \\ &= e_1 e_2^* x + x e_2^* e_1 \\ &= \langle x(\cdot), b_2\rangle\langle a_2, a_1\rangle b_1 + \langle \cdot, a_1\rangle\langle b_1, b_2\rangle x(a_2) \\ &= 0. \end{aligned}$$

This shows $e_1 \,\square\, e_2 = 0$ and, by interchanging the subscripts in the above argument, we also have $e_2 \,\square\, e_1 = 0$. Hence $e_1$ and $e_2$ are triple orthogonal. $\square$

As $V$ is a rank-2 JB*-triple there exist two triple orthogonal minimal tripotents $e_1$ and $e_2$. The joint Peirce decomposition of $V$ with respect to the tripotents $\{e_1, e_2\}$ is given by

$$V \;=\; \bigoplus_{0\le i\le j\le 2} V_{ij}$$

where $V_{ij}$ is the Peirce $ij$-space

$$V_{ij} := \;\; V_{ij}(e_1, e_2) \;\; := \{z \in V \;:\; 2\{e_k, e_k, z\} = (\delta_{ik} + \delta_{jk})z \text{ for } k = 1, 2\}$$

and $\delta_{ij}$ is the Kronecker delta.

More explicitly,

$$\begin{aligned}
V_{22} &= V_2(e_2) = \mathbb{C}e_2 \\
V_{12} &= V_1(e_1) \cap V_1(e_2) \\
V_{02} &= V_0(e_1) \cap V_1(e_2) \\
V_{11} &= V_2(e_1) = \mathbb{C}e_1 \\
V_{01} &= V_1(e_1) \cap V_0(e_2) \\
V_{00} &= V_0(e_1) \cap V_0(e_2).
\end{aligned}$$

The projection from $V$ onto $V_{ij}(e_1, e_2)$ is denoted by $P_{ij}(e_1, e_2)$ or $P_{ij}$. It is sometimes useful to use the following explicit forms of these Peirce projections, which can be derived using rudimentary calculations.

$$\begin{aligned}
P_{22}(e_1, e_2)(T) &= e_2e_2^*Te_2^*e_2 \\
P_{12}(e_1, e_2)(T) &= e_1e_1^*Te_2^*e_2 + e_2e_2^*Te_1^*e_1 \\
P_{02}(e_1, e_2)(T) &= (I_H - e_1e_1^* - e_2e_2^*)Te_2^*e_2 \\
P_{11}(e_1, e_2)(T) &= e_1e_1^*Te_1^*e_1 \\
P_{01}(e_1, e_2)(T) &= (I_H - e_1e_1^* - e_2e_2^*)Te_1^*e_1 \\
P_{00}(e_1, e_2)(T) &= 0
\end{aligned}$$

*Remark* 2.2.3. We observe that the Peirce 00-space of $V$ induced by two triple orthogonal minimal tripotents $e_1$ and $e_2$ is $\{0\}$. This implies that the sum of two triple orthogonal minimal tripotents is a maximal tripotent because $V_0(e_1 + e_2) = V_{00} = \{0\}$ by Theorem 2.1.11.

Another feature of $V$ being a rank-2 JB*-triple is the following particular case of Theorem 2.1.19.

**Corollary 2.2.4.** *For each $x \in V$ there exist triple orthogonal minimal tripotents, $e_1$ and $e_2$, and a uniquely determined element $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha \geq \beta \geq 0$ such that*

$$x = \alpha e_1 + \beta e_2 \tag{2.2.9}$$

*and*

$$\|x\| = \alpha.$$

We will explore more examples of JB*-triples and their connection to bounded symmetric domains in the next section. In fact we will see that $L(\mathbb{C}^2, H)$ is a specific Type I Cartan factor, defined below.

## 2.3 Jordan algebraic classification of bounded symmetric domains

In his seminal work [6], Élie Cartan classified the finite-dimensional irreducible bounded symmetric domains. Although Cartan's classification made use of Lie groups and Lie algebras, it was later found that the classification can be described in terms of JB*-triples. This enables a Jordan approach to the study of bounded symmetric domains, which turned out to be fruitful and have wider applicability to infinite-dimensional domains as well. We discuss briefly Cartan's classification in terms of JB*-triples and list all rank-2 bounded symmetric domains for our investigation of holomorphic dynamics in ensuing chapters.

Let $H$ and $K$ denote complex Hilbert spaces of dimension $n$ and $m$ respectively, where $n, m \in \mathbb{N} \cup \{\infty\}$. The *Cartan factors* come in six series. Let $J : H \to H$ denote an isometric conjugate-linear involution of $H$. For each $x \in L(H)$ define the *transpose of* $x$ as $x^T \in L(H)$ by $x^T = J \circ x^* \circ J$. The following JB*-triples are called Cartan factors.

- Type $I_{n,m}$ : $L(H, K)$ for $1 \leq n < \infty$ and $n \leq m$,
- Type $II_n$ : The closed subtriple $\{z \in L(H) : z^T = -z\}$ of $L(H)$ for $n \geq 2$,
- Type $III_n$ : The closed subtriple $\{z \in L(H) : z^T = z\}$ of $L(H)$,
- Type $IV_n$ : The $n$-dimensional triple spin factor for $n \geq 3$,
- Type $V$ : $M_{12}(\mathcal{O})$, the $1 \times 2$ matrices over the complex octonians, and
- Type $VI$ : $H_3(\mathcal{O})$, the $3 \times 3$ hermitian matrices over the complex octonians.

*Remark* 2.3.1. There are overlaps in the list of Cartan factors, *cf.* [24, p.475; 29, p.48; 32, 4.18]. We have

$$II_2 \cong I_{1,1},\ II_3 \cong I_{1,3},\ II_4 \cong IV_6,\ III_1 \cong I_{1,1},\ IV_3 \cong III_2,\ IV_4 \cong I_{2,2},$$

but this can be avoided by a fastidious choice of the underlying Hilbert space dimension. Indeed each Cartan factor is isometrically isomorphic to one, and only one, of the Cartan factors in the following list, *cf.* [*27*, p.200]:

- $I_{n,m}$ for $1 \leq n < \infty$ and $n \leq m \leq \infty$,
- $II_n$ for $n \geq 5$,
- $III_n$ for $n \geq 2$,
- $IV_n$ for $n \geq 5$,
- $V$, and
- $VI$.

We note that the Cartan factors are JBW*-triples, where $M_{12}(\mathcal{O})$ and $H_3(\mathcal{O})$ are exceptional Jordan triples. We defer discussion of $IV_n$ until Section 4.1 when we review relevant new research on the iteration of holomorphic maps on Lie balls.

Let $\{e_0, e_1, \ldots, e_7\}$ denote a unit basis for the complex Octonians $\mathcal{O}$, with identity element $e_0$. For $x = \sum_{i=0}^{7} w_i e_i \in \mathcal{O}$, we define

$$\begin{aligned} x^* &= \overline{w_0} e_0 - \sum_{i=1}^{7} \overline{w_i} e_i, \\ \tilde{x} &= w_0 e_0 - \sum_{i=1}^{7} w_i e_i, \text{ and} \\ x^\sharp &= \sum_{i=0}^{7} \overline{w_i} e_i. \end{aligned}$$

Now given $A = (a_1, a_2) \in M_{1,2}(\mathcal{O})$ we define $A^*$ by

$$A^* = \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix}.$$

The triple product on $M_{1,2}(\mathcal{O})$ is defined by

$$\{A, B, C\} = \frac{1}{2}\left(A(B^*C) + C(B^*A)\right).$$

The elements of $H_3(\mathcal{O})$ are those $3 \times 3$ matrices over the complex octonians such that $(b_{ij}) = (\tilde{b}_{ji})$. Given $(b_{ij}) \in H_3(\mathcal{O})$ we define $(b_{ij})^* := (b_{ij}^\sharp)$. The triple product on $H_3(\mathcal{O})$ can then be defined by

$$\{A, B, C\} = (A \circ B^*) \circ C + A \circ (B^* \circ C) - B^* \circ (A \circ C),$$

where we have equipped $H_3(\mathcal{O})$ with the Jordan product

$$A \circ B = \frac{1}{2}(AB + BA).$$

The following result of Kaup [26, Theorem 5.1] describes all the bounded symmetric domains in a reflexive Banach space.

**Theorem 2.3.2.** *Each bounded symmetric domain in a reflexive complex Banach space is uniquely (up to order of the factors) biholomorphically equivalent to a direct product $\Omega_1 \times \cdots \times \Omega_k$, where each $\Omega_j$ is the open unit ball of precisely one of the following Jordan triples:*

- $I_{n,m}$ *for* $1 \leq n < \infty$ *and* $n \leq m \leq \infty$,
- $II_n$ *for* $5 \leq n < \infty$,
- $III_n$ *for* $2 \leq n < \infty$,
- $IV_n$ *for* $n \geq 5$,
- $V$, *and*
- $VI$.

**Proposition 2.3.3.** *Each rank-2 bounded symmetric domain is biholomorphically equivalent to either the reducible domain $D_1 \times D_2$, where $D_j$ is the open unit ball of a Hilbert space, or the open unit ball of one of the following rank-2 Cartan factors, which is irreducible:*

- $I_{2,m} = L(\mathbb{C}^2, H)$, where $m = \dim H \geq 2$,
- $II_5 = S_5(\mathbb{C})$, the norm closed subspace of $M_5(\mathbb{C})$ consisting of $5 \times 5$ skew-symmetric matrices,
- $III_2 = H_2(\mathbb{C})$, the norm closed subspace of $M_2(\mathbb{C})$ consisting of $2 \times 2$ symmetric matrices,
- $IV_n = Sp_n(H)$, a complex triple spin factor of dimension $n \geq 5$, and
- $V = M_{12}(\mathcal{O})$, the $1 \times 2$ matrices over the complex octonians.

*Remark* 2.3.4. The only infinite-dimensional rank-2 JB*-triples are $I_{2,\infty}$, $IV_\infty$ and a product of two Hilbert spaces.

# CHAPTER 3

## Holomorphic Dynamics on Products of Hilbert Balls

In this chapter we investigate the dynamics of a holomorphic self-map on a product of Hilbert balls, which are a natural generalisation of the finite-dimensional polydiscs. In Section 3.1, we motivate our discussion with Hervé's classical results on the two-dimensional bidisc [20], followed by a detailed analysis in Sections 3.2 and 3.3 of the case of a finite product of Hilbert balls which can be infinite-dimensional. The main results in the latter case, which generalise the Wolff Theorem (*q.v.* Theorem 1.1.1) and the Denjoy-Wolff Theorem (*q.v.* Theorem 1.1.3), have been published in [12].

## 3.1 Bidisc

The simplest two-dimensional generalisations of the one-dimensional disc in the complex plane are clearly the two-dimensional Euclidean ball and the bidisc. We will introduce Hervé's results [20] for the bidisc in this section. The case of higher dimensional Euclidean balls is subsumed in the study of products of Hilbert balls to be discussed in Sections 3.2 and 3.3.

Hervé [20] has proved the following fundamental result concerning holomorphic iteration on the bidisc.

**Theorem 3.1.1.** *Let $f : D \to D$ be a fixed-point-free holomorphic map on the bidisc $D = \mathbb{D} \times \mathbb{D}$. Then one of the following two conditions holds.*

1. *There exists a point $e^{i\alpha} \in \partial\mathbb{D}$ such that* ***all*** *the limit functions $h$ of $(f^n)$ are of the form*

$$h(x, y) = (e^{i\alpha}, \psi(x, y)),$$

*where $\psi : D \to \overline{\mathbb{D}}$ is a holomorphic map depending on $h$.*

2. *There exists a point $e^{i\beta} \in \partial\mathbb{D}$ such that* ***all*** *the limit functions $\tau$ of $(f^n)$ are of the form*

$$\tau(x, y) = (\varphi(x, y), e^{i\beta}),$$

*where $\varphi : D \to \overline{\mathbb{D}}$ is a holomorphic map depending on $\tau$.*

As Hervé's arguments are intricate and appear to have received insufficient attention in the literature, it would be useful to illuminate, from our perspective and notation, the main ideas and tools Hervé used to prove this important result, and to compare his approach with ours in the case of products of Hilbert balls. First we state the following crucial result [20, Theorem 1, §2, pp2-3], which enabled Hervé to reduce the problem to three mutually exclusive cases that ultimately lead to Theorem 3.1.1.

**Theorem 3.1.2.** *Given a holomorphic function $f : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$, there exists either*

1. *a point $e^{i\alpha} \in \partial\mathbb{D}$ such that, for all $(x, y) \in \mathbb{D}^2$, we have*

$$f(x, y) \in \overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x),$$

*where $\lambda_x = \frac{1-|x|^2}{1-|x|^2+|e^{i\alpha}-x|^2}$ and the boundary of the horosphere $\overline{D}(\lambda_x e^{i\alpha}, 1-\lambda_x)$ contains $x$ and is tangent to $\partial\mathbb{D}$ at $e^{i\alpha}$;*

*or*

2. *a holomorphic function* $\zeta : \mathbb{D} \to \mathbb{D}$*, such that* $f(\zeta(y), y) \equiv \zeta(y)$*.*

*Unless* $f(\cdot, y)$ *is the identity transformation for every* $y$*, these two cases are mutually exclusive and, in the second case,* $f(x, y) = x$ *implies* $x = \zeta(y)$*.*

We note that the function $\ell : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ defined by $\ell(y, x) = f(x, y)$ is also holomorphic. By applying Theorem 3.1.2 to $\ell$ and translating the result for $f$ we have the following corollary.

**Corollary 3.1.3.** *Given a holomorphic function* $f : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$*, there exists either*

1. *a point* $e^{i\beta} \in \partial\mathbb{D}$ *such that, for all* $(x, y) \in \mathbb{D}^2$*, we have*

$$f(x, y) \in \overline{D}(\mu_y e^{i\beta}, 1 - \mu_y),$$

*where* $\mu_y = \frac{1-|y|^2}{1-|y|^2+|e^{i\beta}-y|^2}$ *and the boundary of the horosphere* $\overline{D}(\mu_y e^{i\beta}, 1-\mu_y)$ *contains* $y$ *and is tangent to* $\partial\mathbb{D}$ *at* $e^{i\beta}$*;*

*or*

2. *a holomorphic function* $\nu : \mathbb{D} \to \mathbb{D}$*, such that* $f(x, \nu(x)) \equiv \nu(x)$*.*

*Provided that* $f(x, \cdot)$ *is not the identity transformation for every* $x$*, the two cases are mutually exclusive and, in the second case,* $f(x, y) = y$ *implies* $y = \nu(x)$*.*

**Definition 3.1.4.** Let $\pi_i : \mathbb{C}^2 \to \mathbb{C}$, with $i \in \{1, 2\}$, denote the $i$*-th coordinate map*, that is, $\pi_i(z_1, z_2) = z_i$ for all $(z_1, z_2) \in \mathbb{C}^2$.

A crucial ingredient in Hervé's proof is the fact that every limit function $h$ of $(f^n)$ takes value in the boundary $\partial(\mathbb{D}^2)$ of $\mathbb{D}^2$. This can be stated as follows.

**Proposition 3.1.5.** *If* $f$ *is a fixed-point-free holomorphic self map of the bidisc, then for every* $(u, v) \in \mathbb{D}^2$ *we have*

$$\lim_{n\to\infty} (1 - |\pi_1 \circ f^n(u, v)|)(1 - |\pi_2 \circ f^n(u, v)|) = 0.$$

For the remainder of the section let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a fixed-point-free, holomorphic map. Using coordinate maps set out in Definition 3.1.4 we have $f(x,y) = (\pi_1 \circ f(x,y), \pi_2 \circ f(x,y))$. Hervé initially dealt with the case where $\pi_2 \circ f(x,\cdot)$ is the identity map for every $x$. Then the iteration of $f$ fixes $y$ and, as far as $x$ is concerned, it reduces to the iteration of $\pi_1 \circ f(\cdot, y)$; as $f$ is fixed-point-free, $\pi_1 \circ f(\cdot, y)$ is fixed-point-free for every $y$. Therefore by the Denjoy-Wolff Theorem $\pi_1 \circ f(\cdot, y)$ admits a Wolff point $e^{i\alpha}$, which is independent of $y$. The sequence of iterates of $f$ converge uniformly on compact sets to $h : \mathbb{D} \to \overline{\mathbb{D}}$ defined by $h(x,y) = (e^{i\alpha}, y)$. Therefore, as this case satisfies Theorem 3.1.1, we assume hereinafter, without loss of generality, that neither coordinate remains fixed during iteration of $f$.

Pairing the cases in Theorem 3.1.2 for $\pi_1 \circ f$ and Corollary 3.1.3 for $\pi_2 \circ f$, we have one of the following four cases.

1. There exist

   (a) a point $e^{i\alpha}$ such that, for all $(x,y) \in \mathbb{D}^2$,

   $$\pi_1 \circ f(x,y) \in \overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x),$$

   where $\lambda_x = \frac{1-|x|^2}{1-|x|^2+|e^{i\alpha}-x|^2}$ and the boundary of $\overline{D}(\lambda_x e^{i\alpha}, 1-\lambda_x)$ contains $x$ and is tangent to $\partial\mathbb{D}$ at $e^{i\alpha}$, and

   (b) a holomorphic function $\nu : \mathbb{D} \to \mathbb{D}$, such that $\pi_2 \circ f(x,y) = y$ if and only if $y = \nu(x)$.

2. There exist

   (a) a holomorphic function $\zeta : \mathbb{D} \to \mathbb{D}$, such that $\pi_1 \circ f(x,y) = x$ if and only if $x = \zeta(y)$, and

   (b) a point $e^{i\beta}$ such that, for all $(x,y) \in \mathbb{D}^2$,

   $$\pi_2 \circ f(x,y) \in \overline{D}(\mu_y e^{i\beta}, 1 - \mu_y),$$

where $\mu_y = \frac{1-|y|^2}{1-|y|^2+|e^{i\beta}-y|^2}$ and the boundary of $\overline{D}(\mu_y e^{i\beta}, 1-\mu_y)$ contains $y$ and is tangent to $\partial\mathbb{D}$ at $e^{i\beta}$.

3. There exist

   (a) a point $e^{i\alpha}$ such that, for all $(x,y) \in \mathbb{D}^2$,

$$\pi_1 \circ f(x,y) \in \overline{D}(\lambda_x e^{i\alpha}, 1-\lambda_x),$$

   where $\lambda_x = \frac{1-|x|^2}{1-|x|^2+|e^{i\alpha}-x|^2}$ and the boundary of $\overline{D}(\lambda_x e^{i\alpha}, 1-\lambda_x)$ contains $x$ and is tangent to $\partial\mathbb{D}$ at $e^{i\alpha}$, and

   (b) a point $e^{i\beta}$ such that, for all $(x,y) \in \mathbb{D}^2$,

$$\pi_2 \circ f(x,y) \in \overline{D}(\mu_y e^{i\beta}, 1-\mu_y),$$

   where $\mu_y = \frac{1-|y|^2}{1-|y|^2+|e^{i\beta}-y|^2}$ and the boundary of $\overline{D}(\mu_y e^{i\beta}, 1-\mu_y)$ contains $y$ and is tangent to $\partial\mathbb{D}$ at $e^{i\beta}$.

4. There exist

   (a) a holomorphic function $\zeta : \mathbb{D} \to \mathbb{D}$, such that $\pi_1 \circ f(x,y) = x$ if and only if $x = \zeta(y)$, and

   (b) a holomorphic function $\nu : \mathbb{D} \to \mathbb{D}$, such that $\pi_2 \circ f(x,y) = y$ if and only if $y = \nu(x)$.

The first two cases are not substantively different. Indeed, let $T : \mathbb{D}^2 \to \mathbb{D}^2$ be defined by $T(x,y) = (y,x)$, or equivalently, $T = (\pi_2, \pi_1)$. Now, if $f$ falls into the second case then $\ell : \mathbb{D}^2 \to \mathbb{D}^2$ defined by $\ell = T \circ f \circ T$ falls into the first case and $f^n = (T \circ \ell \circ T)^n = T \circ \ell^n \circ T$. Therefore Theorem 3.1.1 would be proved if we can establish the limit functions take the appropriate given forms in the following three mutually exclusive cases.

1. There exist a point $e^{i\alpha}$ such that, for all $(x, y) \in \mathbb{D}^2$,

$$\pi_1 \circ f(x, y) \in \overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x),$$

where $\lambda_x = \frac{1-|x|^2}{1-|x|^2+|e^{i\alpha}-x|^2}$ and the boundary of $\overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x)$ contains $x$ and is tangent to $\partial\mathbb{D}$ at $e^{i\alpha}$, and a holomorphic function $\eta : \mathbb{D} \to \mathbb{D}$, such that

$$\pi_2 \circ f(x, y) = y \quad \iff \quad y = \eta(x).$$

Then *every* limit function $h$ of $(f^n)$ is of the form

$$h(x, y) = (e^{i\alpha}, \psi(x, y)),$$

for some holomorphic map $\psi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ depending on $h$.

2. There exist two points $e^{i\alpha}$ and $e^{i\beta}$ such that, for all $(x, y) \in \mathbb{D}^2$,

   (a) the point $\pi_1 \circ f(x, y) \in \overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x)$ where $\lambda_x = \frac{1-|x|^2}{1-|x|^2+|e^{i\alpha}-x|^2}$ and the boundary of $\overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x)$ contains $x$ and is tangent to $\partial\mathbb{D}$ at $e^{i\alpha}$, and

   (b) the point $\pi_2 \circ f(x, y) \in \overline{D}(\mu_y e^{i\beta}, 1 - \mu_y)$ where $\mu_y = \frac{1-|y|^2}{1-|y|^2+|e^{i\beta}-y|^2}$ and the boundary of $\overline{D}(\mu_y e^{i\beta}, 1 - \mu_y)$ contains $y$ and is tangent to $\partial\mathbb{D}$ at $e^{i\beta}$.

In this case either *every* limit function $h$ of $(f^n)$ is of the form $h(x, y) = (e^{i\alpha}, \psi(x, y))$, or *every* limit function $h$ of $(f^n)$ is of the form $h(x, y) = (\varphi(x, y), e^{i\beta})$, where the holomorphic functions $\psi$ and $\varphi$ depend on $h$.

3. There exist two holomorphic functions $\xi : \mathbb{D} \to \mathbb{D}$ and $\eta : \mathbb{D} \to \mathbb{D}$ such that

$$\pi_1 \circ f(x, y) = x \quad \iff \quad x = \xi(y)$$
$$\pi_2 \circ f(x, y) = y \quad \iff \quad y = \eta(x).$$

There exist two points $e^{i\alpha}, e^{i\beta}$ such that the sequence $(f^n)$ converges to the constant function $h : \mathbb{D}^2 \to \overline{\mathbb{D}^2}$ defined by $h(\cdot, \cdot) = (e^{i\alpha}, e^{i\beta})$. This third case is the only one in which the sequence $(f^n)$ has a unique limit function.

Hervé [20, §8, p.11] proved the following lemma for use in establishing the form of the limit functions in Cases 1 and 3.

**Lemma 3.1.6.** *In Cases 1 and 3, given $(u, v) \in \mathbb{D}^2$, there exist $a, b > 0$ such that the inequalities*

$$\frac{1 - |\pi_2 \circ f^n(u, v)|}{1 - |\pi_1 \circ f^n(u, v)|} < a \quad \textit{and} \quad \frac{1 - |\pi_2 \circ f^{n+1}(u, v)|}{1 - |\pi_2 \circ f^n(u, v)|} < 1 + b$$

*cannot hold simultaneously.*

We now describe the main tools used in Hervé's analysis of the above three cases.

### 3.1.1 Case 1

Every limit function $h$ takes the form

$$h(x, y) = (e^{i\alpha}, \psi(x, y)).$$

We give a flavour of how this result was achieved.

Let $\mathbb{H} = \{x \in \mathbb{C} \,:\, \mathrm{Re}(x) > 0\}$ be the right-hand half plane of the complex plane and, given $\theta \in [0, 2\pi)$, let $\mathscr{C}_\theta : \mathbb{D} \to \mathbb{H}$ be the Cayley transform defined by

$$\mathscr{C}_\theta(x) = \frac{e^{i\theta} + x}{e^{i\theta} - x}.$$

Clearly the inverse map ${\mathscr{C}_\theta}^{-1} : \mathbb{H} \to \mathbb{D}$ is given by

$${\mathscr{C}_\theta}^{-1}(X) = e^{i\theta} \frac{X - 1}{X + 1}.$$

We define a new function $F : \mathbb{H} \times \mathbb{D} \to \mathbb{H} \times \mathbb{D}$ by

$$F(X, y) \quad = \quad \left( \mathscr{C}_\alpha \circ \pi_1 \circ f({\mathscr{C}_\alpha}^{-1}(X), y) \, , \, \pi_2 \circ f({\mathscr{C}_\alpha}^{-1}(X), y) \right).$$

The family $\left\{ D_x = \overline{D}(\lambda_x e^{i\alpha}, 1 - \lambda_x) \right\}_{x \in \mathbb{D}}$ are the horospheres for $\pi_1 \circ f(\cdot, y)$. As shown in Figure 3.1.1 with $X_i = \mathscr{C}_\alpha(x_i)$, these horospheres $D_{x_i}$ in $\mathbb{D}$ translate into the following right-hand regions in $\mathbb{H}$

$$\{X \in \mathbb{H} \; : \; \mathrm{Re}(X) \geq \mathrm{Re}(X_i)\},$$

and the invariance in these regions is described by

$$\mathrm{Re}(\pi_1 \circ F(X, y)) \quad \geq \quad \mathrm{Re}(X). \tag{3.1.1}$$

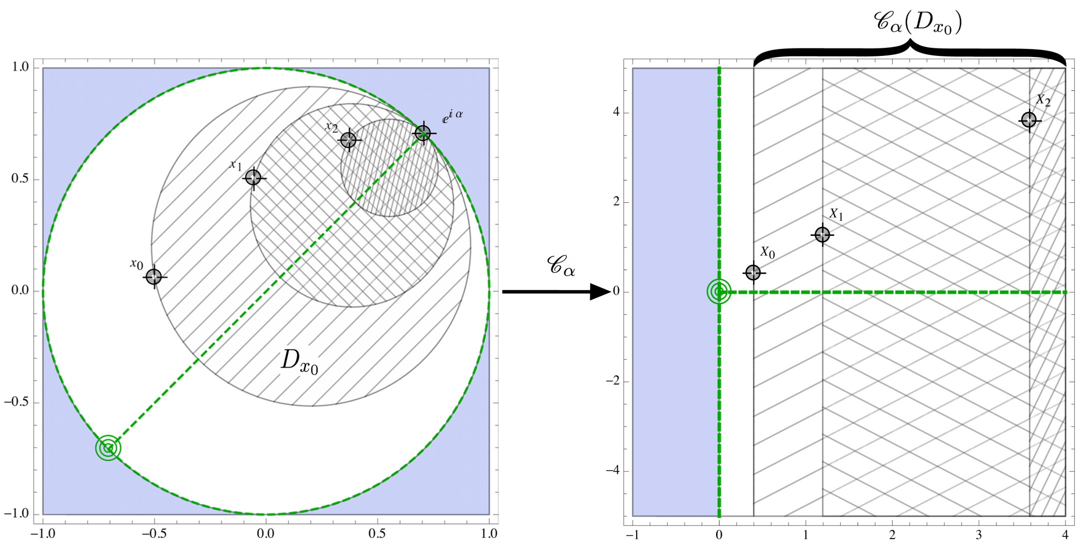



Figure 3.1.1: Corresponding horospheres in the disc and invariant regions in the right-hand half plane.

We recall Harnack's Convergence Theorem, see for example [18, Theorem 2.9].

**Theorem** (Harnack's Convergence Theorem)**.** *Let $(u_n)$ be a sequence of harmonic functions $u_n : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{C}^m$ which satisfy $u_{n+1} \geq u_n$. Then either $u_n \to \infty$ uniformly on compact subsets of $\Omega$ or $(u_n)$ converges to* *a harmonic function on $\Omega$, uniformly on compact subsets of $\Omega$.*

The following calculation shows that the sequence of harmonic functions $(\mathrm{Re} \circ \pi_1 \circ F^n)$ is increasing,

$$\begin{aligned}\mathrm{Re} \circ \pi_1 \circ F^{n+1}(X, y) &= \mathrm{Re} \circ \pi_1 \circ F\Big(\pi_1 \circ F^n(X, y)\,,\, \pi_2 \circ F^n(X, y)\Big) \\ &\geq \mathrm{Re} \circ \pi_1 \circ F^n(X, y) \quad \text{by (3.1.1).}\end{aligned}$$

Now Harnack's Convergence Theorem implies that $(\mathrm{Re} \circ \pi_1 \circ F^n)$ converges uniformly on compacta to either $\infty$ or to a harmonic function $\delta$ that is positive. In the first case, $\mathrm{Re} \circ \pi_1 \circ F^n \to \infty$ implies $\pi_1 \circ f^n \to e^{i\alpha}$. In the other case, Hervé decomposed the function $\pi_1 \circ F^n$ into real and imaginary parts and through some detailed calculations and analysis arrived at the same conclusion. We suppress the details but it would be remiss not to alert the reader that both Proposition 3.1.5 and Lemma 3.1.6 were crucial to the argument.

### 3.1.2 Case 2

In this case, either *every* limit function $h$ is of the form $h(x, y) = (e^{i\alpha}, \psi(x, y))$, or *every* limit function $h$ is of the form $h(x, y) = (\varphi(x, y), e^{i\beta})$. We give an indication of the ingredients used in this proof.

Adopting the same notation as in Section 3.1.1 for the Cayley transform $\mathscr{C}_\theta$ and right-hand half plane $\mathbb{H}$, we consider the function $F : \mathbb{H}^2 \to \mathbb{H}^2$ given by

$$F(X, Y) = \Big(\mathscr{C}_\alpha \circ \pi_1 \circ f(\mathscr{C}_\alpha^{-1}(X), \mathscr{C}_\beta^{-1}(Y))\,,\, \mathscr{C}_\beta \circ \pi_2 \circ f(\mathscr{C}_\alpha^{-1}(X), \mathscr{C}_\beta^{-1}(Y))\Big).$$

As in Case 1, we have the following translation of the invariance of the coordinate horospheres in $\mathbb{D}$ to right-hand regions in $\mathbb{H}$

$$\mathrm{Re}(\pi_1 \circ F(X, Y)) \geq \mathrm{Re}(X)$$

and

$$\mathrm{Re}(\pi_2 \circ F(X,Y)) \;\; \geq \;\; \mathrm{Re}(Y).$$

Therefore $\mathrm{Re}(\pi_1 \circ F^n(X,Y))$ and $\mathrm{Re}(\pi_2 \circ F^n(X,Y))$ form two non-decreasing sequences of harmonic functions. If one or the other tends to $\infty$, then

$$\pi_1 \circ f^n(x,y) \to e^{i\alpha} \quad \text{or} \quad \pi_2 \circ f^n(x,y) \to e^{i\beta}.$$

We suppose therefore that

$$\begin{aligned} \lim_{n\to\infty} \mathrm{Re}(\pi_1 \circ F^n(X,Y)) &= \delta(X,Y) \\ \lim_{n\to\infty} \mathrm{Re}(\pi_2 \circ F^n(X,Y)) &= \delta'(X,Y) \end{aligned}$$

where $\delta, \delta'$ are two positive harmonic functions from $\mathbb{H}^2$ to $\mathbb{R}$.

By invoking the same arguments which we suppressed for Case 1, Hervé showed that

$$\pi_1 \circ F^{n+1}(X,Y) - \pi_1 \circ F^n(X,Y) \;\; \to \;\; i\sigma$$

and

$$\pi_2 \circ F^{n+1}(X,Y) - \pi_2 \circ F^n(X,Y) \;\; \to \;\; i\sigma',$$

where $\sigma$ and $\sigma'$ are constants, and that, if one of them is zero, we have again

$$\pi_1 \circ f^n(x,y) \to e^{i\alpha} \quad \text{or} \quad \pi_2 \circ f^n(x,y) \to e^{i\beta}.$$

Hervé next proved that one of these always had to hold. Indeed, otherwise there would necessarily exist two subsequences $(n_k), (p_k)$ such that

$$\pi_1 \circ f^{n_k}(x,y) \to \varphi(x,y) \qquad \text{and} \qquad \pi_2 \circ f^{p_k}(x,y) \to \psi(x,y)$$

with

$$|\varphi(x,y)| < 1 \quad \text{and} \quad |\psi(x,y)| < 1,$$

which leads to a contradiction when carefully analysing the orbit of the point $(1,1) \in \mathbb{H}^2$ under $F$. We suppress details but note that Hervé's proof implicitly used both the Mean Value Theorem and the Schwarz-Pick Lemma, along with Proposition 3.1.5.

### 3.1.3 Case 3

In this case, Hervé showed the existence of two points $e^{i\alpha}, e^{i\beta}$ such that the sequence $(f^n)$ converges to the constant function $h : \mathbb{D}^2 \to \overline{\mathbb{D}^2}$ defined by $h(x,y) = (e^{i\alpha}, e^{i\beta})$.

Hervé's arguments in this case were protracted and we suppress full details. However, we note that he used Proposition 3.1.5 and Lemma 3.1.6, and now we provide the other two specific lemmas he established [20, Lemmas 1 and 2], to prove the result.

**Lemma 3.1.7.** *There exist two numbers $e^{i\alpha}, e^{i\beta} \in \partial\mathbb{D}$ and two positive numbers $\lambda, \mu$ such that if $\rho, \rho' \in (-1,1)$ satisfy*

$$\lambda \frac{1+\rho}{1-\rho} = \mu \frac{1+\rho'}{1-\rho'}, \tag{3.1.2}$$

*then* $f\left(\overline{D}(\frac{1+\rho}{2}e^{i\alpha}, \frac{1-\rho}{2}) \times \overline{D}(\frac{1+\rho'}{2}e^{i\beta}, \frac{1-\rho'}{2})\right) \subset \overline{D}(\frac{1+\rho}{2}e^{i\alpha}, \frac{1-\rho}{2}) \times \overline{D}(\frac{1+\rho'}{2}e^{i\beta}, \frac{1-\rho'}{2})$.

Choosing the points $e^{i\alpha}, e^{i\beta}, \lambda, \mu$ in line with Lemma 3.1.7, once again we adopt the same notation as before for the Cayley transform $\mathscr{C}_\theta$ and right-hand half plane $\mathbb{H}$, and akin to Section 3.1.2 we consider the function $F : \mathbb{H}^2 \to \mathbb{H}^2$ given by

$$F(X,Y) = \Big( \mathscr{C}_\alpha \circ \pi_1 \circ f(\mathscr{C}_\alpha^{-1}(X), \mathscr{C}_\beta^{-1}(Y)) \, , \mathscr{C}_\beta \circ \pi_2 \circ f(\mathscr{C}_\alpha^{-1}(X), \mathscr{C}_\beta^{-1}(Y)) \Big).$$

We note that (3.1.2) encodes the relationship $\lambda \mathscr{C}_\alpha(\rho e^{i\alpha}) = \mu \mathscr{C}_\beta(\rho' e^{i\beta})$, where $\rho e^{i\alpha}$ and $\rho' e^{i\beta}$ are boundary points of the discs $\overline{D}(\frac{1+\rho}{2}e^{i\alpha}, \frac{1-\rho}{2})$ and $\overline{D}(\frac{1+\rho'}{2}e^{i\beta}, \frac{1-\rho'}{2})$ respectively. Moreover the images in the right-hand half plane $\mathbb{H}$ of the discs $\overline{D}(\frac{1+\rho}{2}e^{i\alpha}, \frac{1-\rho}{2})$ and $\overline{D}(\frac{1+\rho'}{2}e^{i\beta}, \frac{1-\rho'}{2})$ , under the appropriate Cayley transform, are the regions

$$\{X \in \mathbb{H} \, : \, \mathrm{Re}(X) \geq c_1\} \quad \text{and} \quad \{Y \in \mathbb{H} \, : \, \mathrm{Re}(Y) \geq c_2\}$$

respectively, where $c_1 = \mathrm{Re}(\mathscr{C}_\alpha(\rho e^{i\alpha}))$ and $c_2 = \mathrm{Re}(\mathscr{C}_\alpha(\rho' e^{i\alpha}))$ are constants. In fact these constants are inversely proportional to $\lambda$ and $\mu$ respectively because

$$\begin{aligned} \mathrm{Re}(\mathscr{C}_\alpha(\rho e^{i\alpha})) = &\ \mathrm{Re}\left(\tfrac{e^{i\alpha}+\rho e^{i\alpha}}{e^{i\alpha}-\rho e^{i\alpha}}\right) = \frac{1+\rho}{1-\rho} = \frac{\mu}{\lambda}\frac{1+\rho'}{1-\rho'} \propto \frac{1}{\lambda}, \\ \mathrm{Re}(\mathscr{C}_\alpha(\rho' e^{i\alpha})) = &\ \mathrm{Re}\left(\tfrac{e^{i\beta}+\rho' e^{i\beta}}{e^{i\beta}-\rho' e^{i\beta}}\right) = \frac{1+\rho'}{1-\rho'} = \frac{\lambda}{\mu}\frac{1+\rho}{1-\rho} \propto \frac{1}{\mu}. \end{aligned}$$

Given $(X,Y) \in \mathbb{H}^2$, we can assume, without loss of generality, that $\mathscr{C}_\alpha^{-1}(X)$ is on the boundary of the disc $\overline{D}(\frac{1+\rho}{2}e^{i\alpha}, \frac{1-\rho}{2})$. Therefore

$$\lambda \mathrm{Re}(X) = \lambda \frac{1+\rho}{1-\rho} = \mu \frac{1+\rho'}{1-\rho'} \leq \mu \mathrm{Re}(Y)$$

and Lemma 3.1.7 implies that

$$\begin{aligned} \lambda \mathrm{Re}(\pi_1 \circ F(X,Y)) &\geq \lambda \mathrm{Re}(X), \\ \mu \mathrm{Re}(\pi_2 \circ F(X,Y)) &\geq \mu \frac{1+\rho'}{1-\rho'} = \lambda \frac{1+\rho}{1-\rho} = \lambda \mathrm{Re}(X). \end{aligned}$$

Hence the invariance in Lemma 3.1.7 translates to $\mathbb{H}^2$ through the inequality

$$\min\{\lambda \mathrm{Re}(\pi_1 \circ F(X,Y))\,,\, \mu \mathrm{Re}(\pi_2 \circ F(X,Y))\} \;\geq\; \min\{\lambda \mathrm{Re}(X), \mu \mathrm{Re}(Y)\}.$$

Therefore the sequence $(\min\{\lambda \mathrm{Re}(\pi_1 \circ F^n(X,Y))\,,\, \mu \mathrm{Re}(\pi_2 \circ F^n(X,Y))\})$ is non-decreasing and converges to some limit $\delta(X,Y)$.

**Lemma 3.1.8.** *At each point $(u,v) \in \mathbb{D}^2$ one can associate a number $c(u,v) > 0$ such that, for $m$ which satisfies the condition $\frac{1-|\pi_2 \circ f^m(u,v)|}{1-|\pi_1 \circ f^m(u,v)|} < c$, we have*

$$\lambda Re\,(\pi_1 \circ F^m(\mathscr{C}_\alpha(u), \mathscr{C}_\beta(v))) \;\to\; \delta(\mathscr{C}_\alpha(u), \mathscr{C}_\beta(v)) \quad \textit{as } m \to \infty.$$

It is evident now that Hervé's ingenious arguments are hard to extend directly to the product of two higher dimensional Euclidean balls, let alone the infinite-dimensional ones. In what follows, we will adopt a different approach to the latter case, using Jordan geometry.

## 3.2 Invariant domains and horospheres

In this section and the next, we explore the iteration of a holomorphic self-map on a finite Cartesian product of Hilbert balls. This work has been published in [12]. Let $p$ be a fixed positive integer and for each $j \in \{1, \ldots, p\}$ let $V_j$ be a Hilbert space with open unit ball $D_j$. For the remainder of the chapter, $V$ is the $\ell_\infty$-sum of the $p$ Hilbert spaces $V_1, \ldots, V_p$ :

$$V = V_1 \oplus_\infty \cdots \oplus_\infty V_p.$$

We will use the relevant notation and results given in Section 2.2. For instance, we see from the discussion in Section 2.2 that $V$ admits the structure of a JB*-triple, which we will assume henceforth. The open unit ball of $V$ is the polyball $D = D_1 \times \cdots \times D_p$. The topological boundaries of $D$ and $D_j$ are denoted by

$\partial D = \{\boldsymbol{z} \in V : \|\boldsymbol{z}\| = 1\}$ and $\partial D_j = \{z \in V_j : \|z\| = 1\}$ respectively. For a set $E \subset V$ we denote the norm closure by $\overline{E}$. The weak topology on $D$ is the product of the weak topologies of $D_1, \ldots, D_{p-1}$ and $D_p$. Recall that a holomorphic map $f : D \longrightarrow D$ is Kobayashi nonexpansive by the Schwarz-Pick Lemma.

**Lemma 3.2.1.** *Given $\boldsymbol{b} = (b_1, \ldots, b_p)$ and $\boldsymbol{c} = (c_1, \ldots, c_p)$ in the closure of $D = D_1 \times \cdots \times D_p$, we have $B(\boldsymbol{b}, \boldsymbol{c}) = 0$ if and only if $b_j = c_j \in \partial D_j$ for all $j = 1, \ldots, p$.*

*Proof.* This follows from the fact that $B(\boldsymbol{b}, \boldsymbol{c}) = 0$ if and only if $B(b_j, c_j) = 0$ for $j = 1, \ldots, p$, where

$$B(b_j, c_j)(x_j) = x_j - \langle b_j, c_j \rangle_j x_j - \langle x_j, c_j \rangle_j b_j + \langle x_j, c_j \rangle_j \langle b_j, c_j \rangle_j b_j \qquad (x_j \in D_j)$$

and $B(b_j, c_j)(b_j) = (1 - \langle b_j, c_j \rangle_j)^2 b_j$ imply that $B(b_j, c_j) = 0$ if and only if $\langle b_j, c_j \rangle_j = 1$ which is equivalent to $b_j = c_j \in \partial D_j$. $\square$

**Lemma 3.2.2.** *Let $D = D_1 \times \cdots \times D_p$ be a polyball. Given a sequence $(\boldsymbol{a}_k)$ in $D$ norm converging to $\boldsymbol{a} = (a_1, \ldots, a_p) \in \partial D$ and a sequence $(\boldsymbol{v}_k)$ in $D$ weakly convergent to some $\boldsymbol{v} = (v_1, \ldots, v_p) \in D$, we have*

$$\lim_{k \to \infty} \|g_{-\boldsymbol{a}_k}(\boldsymbol{v}_k)\| = 1.$$

*Proof.* Write $\boldsymbol{a}_k = (a_{k1}, \ldots, a_{kp})$ and $\boldsymbol{v}_k = (v_{k1}, \ldots, v_{kp})$. Since $a_j \in \partial D_j$ for some $j \in \{1, \ldots, p\}$, we have from (2.2.1) that

$$\|B(a_{kj}, a_{kj})^{1/2}(v_{kj})\| \leq \sqrt{1 - \|a_{kj}\|^2} \left( 2 - \sqrt{1 - \|a_{kj}\|^2)} \right) \longrightarrow 0 \quad \text{as } k \to \infty$$

which gives, using (2.2.3),

$$
\begin{aligned}
1 \quad > \quad & \|g_{-\boldsymbol{a}_k}(\boldsymbol{v}_k)\| \geq \|g_{-a_{kj}}(v_{kj})\| = \left\| -a_{kj} + \frac{1}{1-\langle v_{kj}, a_{kj}\rangle} B(a_{kj}, a_{kj})^{1/2}(v_{kj}) \right\| \\
\geq \quad & \|a_{kj}\| - \frac{1}{|1-\langle v_{kj}, a_{kj}\rangle|} \|B(a_{kj}, a_{kj})^{1/2}(v_{kj})\| \longrightarrow 1, \quad \text{as} \quad k \to \infty
\end{aligned}
$$

where $\langle v_j, a_j \rangle \neq 1$. □

**Lemma 3.2.3.** *Let $(\boldsymbol{a}_k)$ and $(\boldsymbol{b}_k)$ be two sequences in $D = D_1 \times \cdots \times D_p$ converging to two boundary points $\boldsymbol{a} = (a_1, \ldots, a_p)$ and $\boldsymbol{b} = (b_1, \ldots, b_p)$ respectively. If $a_j \neq b_j$ and $\|a_j\| = \|b_j\| = 1$ for some $j \in \{1, \ldots, p\}$, then we have*

$$\lim_{k\to\infty} \|g_{-\boldsymbol{b}_k}(\boldsymbol{a}_k)\| = 1.$$

*Proof.* Write $\boldsymbol{a}_k = (a_{k1}, \ldots, a_{kp})$ and $\boldsymbol{b}_k = (b_{k1}, \ldots, b_{kp})$. We first observe that

$$
\begin{aligned}
&\|B(\boldsymbol{a}_k, \boldsymbol{a}_k)^{-1/2} B(\boldsymbol{a}_k, \boldsymbol{b}_k) B(\boldsymbol{b}_k, \boldsymbol{b}_k)^{-1/2}\| \\
&\geq \|B(a_{kj}, a_{kj})^{-1/2} B(a_{kj}, b_{kj}) B(b_{kj}, b_{kj})^{-1/2}\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\|B(a_{kj}, b_{kj})\| \\
&= \|B(a_{kj}, a_{kj})^{1/2} B(a_{kj}, a_{kj})^{-1/2} B(a_{kj}, b_{kj}) B(b_{kj}, b_{kj})^{-1/2} B(b_{kj}, b_{kj})^{1/2}\| \\
&\leq \|B(a_{kj}, a_{kj})^{1/2}\| \|B(a_{kj}, a_{kj})^{-1/2} B(a_{kj}, b_{kj}) B(b_{kj}, b_{kj})^{-1/2}\| \|B(b_{kj}, b_{kj})^{1/2}\|.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
1 - \|g_{-\boldsymbol{b}_k}(\boldsymbol{a}_k)\|^2 \quad &= \quad \|B(\boldsymbol{a}_k, \boldsymbol{a}_k)^{-1/2} B(\boldsymbol{a}_k, \boldsymbol{b}_k) B(\boldsymbol{b}_k, \boldsymbol{b}_k)^{-1/2}\|^{-1} \\
&\leq \quad \|B(a_{kj}, a_{kj})^{-1/2} B(a_{kj}, b_{kj}) B(b_{kj}, b_{kj})^{-1/2}\|^{-1} \\
&\leq \quad \frac{\|B(a_{kj}, a_{kj})^{1/2}\| \|B(b_{kj}, b_{kj})^{1/2}\|}{\|B(a_{kj}, b_{kj})\|} \longrightarrow 0 \quad \text{as} \quad k \to \infty,
\end{aligned}
$$

where $\lim\limits_k B(a_{kj}, b_{kj}) = B(a_j, b_j) \neq 0$ by Lemma 3.2.1 while $\lim\limits_k B(a_{kj}, a_{kj}) = B(a_j, a_j) = 0$ and $B(b_j, b_j) = 0$. □

Given a fixed-point-free holomorphic map $f : D \longrightarrow D$, we study the asymptotic behaviour of the iterates $(f^n)$. As in the case of the complex unit disc, we first study the invariant domains of $f$. These invariant domains, often called horospheres, can be defined by sequences of Kobayashi balls as in [2, 3, 23, 34]. We adopt this approach for the polyballs and give an explicit description of the horospheres in terms of the Bergmann operators.

Fix a holomorphic map $f : D \longrightarrow D$ without fixed point. Choose an increasing sequence $(\alpha_k)$ in $(0, 1)$ with limit 1. Then $\alpha_k f$ maps $D$ strictly inside itself and we have $\alpha_k f(\boldsymbol{z}_k) = \boldsymbol{z}_k$ for some $\boldsymbol{z}_k \in D$ by the Earle-Hamilton Fixed Point Theorem [16].

For a single Hilbert ball $D = D_1$, We may assume, by choosing a subsequence if necessary, that $(\boldsymbol{z}_k)$ converges weakly to some point $\boldsymbol{\xi} \in \overline{D}$. Then we have $\boldsymbol{\xi} \in \partial D$. Indeed, suppose to the contrary that $\boldsymbol{\xi} \in D$, then we have

$$\|g_{-f(\boldsymbol{\xi})}(f(\boldsymbol{z}_k))\| \leq \|g_{-\boldsymbol{\xi}}(\boldsymbol{z}_k)\|.$$

Using (2.2.5) and substituting $f(\boldsymbol{z}_k) = \boldsymbol{z}_k/\alpha_k$, one gets, analogous to [38, 8.1.4(2)], that

$$\frac{1 - \alpha_k^{-2}\|\boldsymbol{z}_k\|^2}{1 - \|\boldsymbol{z}_k\|^2} \frac{|1 - \langle \boldsymbol{z}_k, \boldsymbol{\xi}\rangle|^2}{1 - \|\boldsymbol{\xi}\|^2} \geq \frac{|1 - \langle \alpha_k^{-1}\boldsymbol{z}_k, f(\boldsymbol{\xi})\rangle|^2}{1 - \|f(\boldsymbol{\xi})\|^2}.$$

Since $\frac{1-\alpha_k^{-2}\|\boldsymbol{z}_k\|^2}{1-\|\boldsymbol{z}_k\|^2} \leq 1$, we have

$$1 \geq \frac{1}{1 - \|g_{-f(\boldsymbol{\xi})}(\boldsymbol{\xi}))\|^2}$$

by letting $k \to \infty$. This gives $\|g_{-f(\boldsymbol{\xi})}(\boldsymbol{\xi}))\| = 0$ and $f(\boldsymbol{\xi}) = \boldsymbol{\xi}$, contradicting the non-existence of a fixed point in $D$. It follows that $(\boldsymbol{z}_k)$ actually norm converges to $\boldsymbol{\xi}$ since $\limsup_{k\to\infty} \|\boldsymbol{z}_k\| \leq 1 = \|\boldsymbol{\xi}\| \leq \liminf_{k\to\infty} \|\boldsymbol{z}_k\|$.

For a polyball $D = D_1 \times \cdots \times D_p$ with $p \geq 2$, we assume throughout that $f$ is *compact*, that is, the closure $\overline{f(D)}$ is compact in $\overline{D}$. In this case, the sequence $(\alpha_k^{-1}\boldsymbol{z}_k)$ lies in a compact set and, passing to a subsequence if necessary, we may assume $(\boldsymbol{z}_k)$ norm converges to some $\boldsymbol{\xi} \in \overline{D}$. Since $f$ has no fixed point, we also have $\boldsymbol{\xi} \in \partial D$.

From now on, let $p \in \mathbb{N} = \{1, 2, 3, \dots\}$ and fix the sequence $(\boldsymbol{z}_k)$ and $\boldsymbol{\xi} \in \partial D$, which depend on $f$. We denote by

$$D(\boldsymbol{v}, r) = \{\boldsymbol{z} \in V : \|\boldsymbol{z} - \boldsymbol{v}\| < r\}$$

the open ball centred at $\boldsymbol{v} \in V$, of radius $r > 0$.

Given any $\lambda > 0$, we can find a sequence $(s_k)$ in $(0, 1)$ such that

$$\lambda = \frac{1 - s_k^2}{1 - \|\boldsymbol{z}_k\|^2}$$

from some $k$ onwards and hence, by discarding only finitely many initial terms, we may assume without loss of generality the same holds for all $k$. The convergence of $(\|\boldsymbol{z_k}\|)$ to 1 necessitates the same of $(s_k)$. We define a (closed) Kobayashi ball centre at $\boldsymbol{z}_k$ to be the set

$$D_k[\lambda] := g_{\boldsymbol{z}_k}(\overline{D}(\mathbf{0}, s_k)).$$

Following [3], we define a *closed horosphere* $S(\boldsymbol{\xi}, \lambda)$ *at* $\boldsymbol{\xi}$ as the limit of a sequence of Kobayashi balls as follows:

$$S(\boldsymbol{\xi}, \lambda) := \{\boldsymbol{x} \in \overline{D} : \boldsymbol{x} = \lim_k \boldsymbol{x}_k \text{ and } \boldsymbol{x}_k \in D_k[\lambda]\}$$

where the limit is taken in the norm.

The intersection $S(\boldsymbol{\xi}, \lambda) \cap D$ is called a *horosphere* in $D$. We note that $\boldsymbol{z}_k \in D_k[\lambda]$ and $\boldsymbol{\xi} \in S(\boldsymbol{\xi}, \lambda)$. It should be noted that if $(r_k)$ is another sequence in $(0, 1)$, with limit 1, and satisfies $\lambda = \lim_k \dfrac{1 - r_k^2}{1 - \|\boldsymbol{z}_k\|^2}$, then the Kobayashi balls $g_{\boldsymbol{z}_k}(\overline{D}(\mathbf{0}, s_k))$ and $g_{\boldsymbol{z}_k}(\overline{D}(\mathbf{0}, r_k))$ would produce the same closed horosphere $S(\boldsymbol{\xi}, \lambda)$. Hence the construction of $S(\boldsymbol{\xi}, \lambda)$ does not depend on the choice of the sequence $(s_k)$.

We will see that $S(\boldsymbol{\xi}, \lambda)$ is weakly compact in $V$ and convex, and is exactly the closure of $S(\boldsymbol{\xi}, \lambda) \cap D$. These horospheres enable one to obtain in Proposition 3.2.5 below a generalised version of Wolff's Theorem for a fixed-point-free holomorphic map $f : \mathbb{D} \longrightarrow \mathbb{D}$, which asserts the existence of a family of $f$-invariant discs $S_\lambda$ of radii $\lambda > 0$ at a boundary point $\xi \in \partial\mathbb{D}$ (*q.v.* Theorem 1.1.1).

*Remark* 3.2.4. We note that $S(\boldsymbol{\xi}, \lambda)$ depends on the sequence $(\boldsymbol{z}_k)$. To simplify notation, we suppress explicit reference to $(\boldsymbol{z}_k)$.

**Proposition 3.2.5.** *Let $D = D_1 \times \cdots \times D_p$ and $f : D \longrightarrow D$ be a fixed-point-free holomorphic map, which is compact if $p \geq 2$. Then there is a sequence $(\boldsymbol{z}_k)$ in $D$ converging to a boundary point $\boldsymbol{\xi} \in \partial D$ such that, for all $\lambda > 0$, we have $f(S(\boldsymbol{\xi}, \lambda) \cap D) \subset S(\boldsymbol{\xi}, \lambda) \cap D$.*

*Proof.* Let $(\boldsymbol{z}_k)$ be the sequence constructed above, with limit $\boldsymbol{\xi} \in \partial D$, and let $f_k = \alpha_k f$ which is Kobayashi nonexpansive.

Given any $\boldsymbol{x} \in S(\boldsymbol{\xi}, \lambda) \cap D$ with $\boldsymbol{x} = \lim_k \boldsymbol{x}_k$ and $\boldsymbol{x}_k \in D_k[\lambda]$, we have

$$\|g_{-\boldsymbol{z}_k}(f_k(\boldsymbol{x}_k))\| = \|g_{-f_k(\boldsymbol{z}_k)}(f_k(\boldsymbol{x}_k))\| \leq \|g_{-\boldsymbol{z}_k}(\boldsymbol{x}_k)\| \leq s_k$$

and hence $f_k(\boldsymbol{x}_k) \in D_k[\lambda]$. It follows that $f(\boldsymbol{x}) = \lim_k f(\boldsymbol{x}_k) = \lim_k f_k(\boldsymbol{x}_k) \in S(\boldsymbol{\xi}, \lambda)$. □

*Remark* 3.2.6. The above construction of the horosphere $S(\boldsymbol{\xi}, \lambda)$, as well as the invariance $f(S(\boldsymbol{\xi}, \lambda) \cap D) \subset S(\boldsymbol{\xi}, \lambda) \cap D$, are actually valid for the open unit ball $D$ of any JB*-triple.

We now give a more explicit description of $S(\boldsymbol{\xi}, \lambda)$ for later applications. For $\boldsymbol{z} \in D$, using the formula

$$g_{\boldsymbol{z}}(r\boldsymbol{x}) = (1 - r^2)B(r\boldsymbol{z}, r\boldsymbol{z})^{-1/2}(\boldsymbol{z}) + rB(\boldsymbol{z}, \boldsymbol{z})^{1/2}B(r\boldsymbol{z}, r\boldsymbol{z})^{-1/2}g_{r\boldsymbol{z}}(\boldsymbol{x}) \quad (3.2.1)$$

for $\boldsymbol{x} \in \overline{D}$ and $0 < r < 1$, it has been shown in [34, Proposition 2.3] that the (open) Kobayashi ball $g_{\boldsymbol{z}_k}(D(0, s_k))$ has the form

$$g_{\boldsymbol{z}_k}(D(0, s_k)) = \boldsymbol{c}_k(\lambda) + s_k B(\boldsymbol{z}_k, \boldsymbol{z}_k)^{1/2} B(s_k \boldsymbol{z}_k, s_k \boldsymbol{z}_k)^{-1/2}(D) \tag{3.2.2}$$

where

$$\boldsymbol{c}_k(\lambda) = (1 - s_k^2) B(s_k \boldsymbol{z}_k, s_k \boldsymbol{z}_k)^{-1/2}(\boldsymbol{z}_k).$$

Likewise, the above formula for $g_{\boldsymbol{z}}(r\boldsymbol{x})$ gives

$$D_k[\lambda] = g_{\boldsymbol{z}_k}(\overline{D}(0, s_k)) = \boldsymbol{c}_k(\lambda) + s_k B(\boldsymbol{z}_k, \boldsymbol{z}_k)^{1/2} B(s_k \boldsymbol{z}_k, s_k \boldsymbol{z}_k)^{-1/2}(\overline{D}) \tag{3.2.3}$$

where we will write $\boldsymbol{c}_k(\lambda) = (\boldsymbol{c}_k(\lambda)_1, \ldots, \boldsymbol{c}_k(\lambda)_p)$ and $\boldsymbol{z}_k = (z_{k1}, \ldots, z_{kp})$.

From (2.2.2) and linearity of the Bergmann operators, we deduce

$$\begin{aligned} \boldsymbol{c}_k(\lambda)_j &= (1 - s_k^2) B(s_k z_{kj}, s_k z_{kj})^{-1/2}(z_{kj}) \\ &= \frac{1 - s_k^2}{1 - s_k^2 \|z_{kj}\|^2} z_{kj}. \end{aligned}$$

For $j \in \{1, \ldots, p\}$, let $t_{kj}^2 = \frac{1-s_k^2}{1-s_k^2\|z_{kj}\|^2}$ for $k = 1, 2, \ldots$. Then $0 < t_{kj}^2 < 1$ implies

$$\limsup_{k\to\infty} \frac{1 - s_k^2}{1 - s_k^2 \|z_{kj}\|^2} = t_j^2$$

for some $t_j \in [0, 1]$. Likewise $\tau_k^2 = \frac{1-s_k^2}{1-s_k^2\|\boldsymbol{z}_k\|^2} \in (0, 1)$ implies

$$\limsup_{k\to\infty} \tau_k^2 = \tau^2$$

for some $\tau \in [0, 1]$. Hereinafter, when there is no loss of generality, we may assume, by choosing a subsequence if necessary, that the sequences $(t_{kj}^2)_k$ and $(\tau_k^2)$ converge to $t_j^2$ and $\tau^2$ respectively, for $j \in \{1, \ldots, p\}$.

Observe that

$$\frac{1-s_k^2}{1-s_k^2\|\boldsymbol{z}_k\|^2} = \frac{1-s_k^2}{1-s_k^2+s_k^2(1-\|\boldsymbol{z}_k\|^2)} = \frac{\lambda}{\lambda+s_k^2}.$$

Therefore $\tau^2 = \limsup_{k\to\infty} \tau_k^2 = \dfrac{\lambda}{\lambda+1} > 0$.

Since $\|\boldsymbol{z}_k\| = \sup\{\|z_{kj}\| : j = 1,\ldots,p\}$, there are two possibilities for each $j \in \{1,\ldots,p\}$, namely, either $\|z_{kj}\| < \|\boldsymbol{z}_k\|$ from some $k$ onwards or, $\|z_{kj}\| = \|\boldsymbol{z}_k\|$ frequently and there is a subsequence $(z_{k'j})$ of $(z_{kj})$ with $\|z_{k'j}\| = \|\boldsymbol{z}_{k'}\|$. Since $\{1,\ldots,p\}$ is finite, the latter must occur for some $j$ in which case, we have

$$t_j^2 = \lim_{k'} t_{k'j}^2 = \lim_{k'} \tau_{k'}^2 > 0.$$

Although $t_j$ above depends on the parameter $\lambda > 0$, its positivity is dependent only on the sequence $(\boldsymbol{z}_k)$. More precisely, we have the following lemma.

**Lemma 3.2.7.** *Let $t_j$ be defined as above. Then for each $j \in \{1,\ldots,p\}$ we have $t_j > 0$ if and only if $\liminf_{k\to\infty} \dfrac{1-\|z_{kj}\|^2}{1-\|\boldsymbol{z}_k\|^2} < \infty$.*

*Proof.* From the calculation

$$\begin{aligned}\frac{1-s_k^2}{1-s_k^2\|z_{kj}\|^2} &= \frac{1-s_k^2}{1-s_k^2+s_k^2(1-\|z_{kj}\|^2)} \\ &= \frac{\lambda}{\lambda+s_k^2\frac{1-\|z_{kj}\|^2}{1-\|\boldsymbol{z}_k\|^2}},\end{aligned}$$

one sees that $t_j^2 = \limsup_{k\to\infty} \dfrac{1-s_k^2}{1-s_k^2\|z_{kj}\|^2} > 0$ if and only if $\liminf_{k\to\infty} \dfrac{1-\|z_{kj}\|^2}{1-\|\boldsymbol{z}_k\|^2} < \infty$. □

Let $J = \{j \in \{1,\ldots,p\} : t_j > 0\}$. Then $J \neq \emptyset$ and $J$ does not depend on the parameter $\lambda > 0$. Moreover, Lemma 3.2.7 gives

$$J = \left\{ j \in \{1,\ldots,p\} : \gamma_j = \liminf_{k\to\infty} \frac{1-\|z_{kj}\|^2}{1-\|\boldsymbol{z}_k\|^2} < \infty \right\}, \tag{3.2.4}$$

where $\gamma_j \geq 1$. For every $j \in J$ we have $t_j = \sqrt{\frac{\lambda}{\lambda+\gamma_j}} \in (0,1)$.

**Proposition 3.2.8.** *Let $D = D_1 \times \cdots \times D_p$ be a polyball and $f : D \longrightarrow D$ be a fixed-point-free holomorphic map, which is compact if $p \geq 2$. Then there exist a sequence $(\boldsymbol{z}_k)$ in $D$ converging to a boundary point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p) \in \partial D$, and a nonempty set $J \subset \{1, \ldots, p\}$, such that for any $\lambda > 0$, we have $S(\boldsymbol{\xi}, \lambda) = S_1(\xi_1, \lambda) \times \cdots \times S_p(\xi_p, \lambda)$ where*

$$S_j(\xi_j, \lambda) = \begin{cases} t_j^2\, \xi_j + B(t_j\, \xi_j, t_j\, \xi_j)^{1/2}(\overline{D}_j) & (j \in J) \\ \\ \overline{D}_j & (j \notin J) \end{cases}$$

*and $t_j = \sqrt{\frac{\lambda}{\lambda + \gamma_j}} \in (0,1)$ where $\gamma_j = \liminf_{k \to \infty} \frac{1 - \|z_{kj}\|^2}{1 - \|\boldsymbol{z}_k\|^2} < \infty$.*

*Proof.* Let $(\boldsymbol{z}_k)$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$ be defined as before. Let $J \subset \{1, \ldots, p\}$ be the nonempty set defined in (3.2.4). Let $\boldsymbol{x} = (x_1, \ldots, x_p) \in S(\boldsymbol{\xi}, \lambda)$ with $\boldsymbol{x} = \lim_k \boldsymbol{x}_k$ and $\boldsymbol{x}_k = (x_{k1}, \ldots, x_{kp}) \in D_k[\lambda]$. For each $j \in \{1, \ldots, p\}$, we have

$$\begin{aligned} x_{kj} &= \boldsymbol{c}_k(\lambda)_j + s_k B(z_{kj}, z_{kj})^{1/2} B(s_k z_{kj}, s_k z_{kj})^{-1/2}(w_{kj}) \\ &= t_{kj}^2\, z_{kj} + s_k B(z_{kj}, z_{kj})^{1/2} B(s_k z_{kj}, s_k z_{kj})^{-1/2}(w_{kj}) \end{aligned}$$

for some $w_{kj} \in \overline{D}_j$. By weak compactness and passing to a subsequence if necessary, we may assume that $(w_{kj})$ weakly converges to some $w_j \in \overline{D}_j$.

Let $j \in \{1, \ldots, p\}$. If $\|\xi_j\| = \lim_k \|z_{kj}\| < 1$, then $t_j^2 = \lim_k t_{kj}^2 = 0$ and we have $x_j = \lim_k x_{kj} = w_j \in \overline{D}_j$.

Let $\|\xi_j\| = 1$. By [9, Example 3.2.5], for $a \in D_j \backslash \{0\}$ we have

$$B(a,a)^{-1/2}(z) = \frac{1}{\sqrt{1 - \|a\|^2}} \left( z + \frac{1 - \sqrt{1 - \|a\|^2}}{\|a\|^2 \sqrt{1 - \|a\|^2}} \langle z, a \rangle a \right) \qquad (z \in D_j). \tag{3.2.5}$$

From (2.2.1) and (3.2.5), we obtain

$$\begin{aligned} & B(z_{kj}, z_{kj})^{1/2} B(s_k z_{kj}, s_k z_{kj})^{-1/2}(w_{kj}) \\ = & \sqrt{\frac{1 - \|z_{kj}\|^2}{1 - s_k^2 \|z_{kj}\|^2}}\, w_{kj} + \left( \frac{1 - \|z_{kj}\|^2}{1 - s_k^2 \|z_{kj}\|^2} - \sqrt{\frac{1 - \|z_{kj}\|^2}{1 - s_k^2 \|z_{kj}\|^2}} \right) \frac{\langle w_k, z_{kj} \rangle z_{kj}}{\|z_{kj}\|^2} \end{aligned}$$

where

$$\frac{1-\|z_{kj}\|^2}{1-s_k^2\|z_{kj}\|^2} = \frac{1}{s_k^2}\left(1-\frac{1-s_k^2}{1-s_k^2\|z_{kj}\|^2}\right) \longrightarrow 1-t_j^2 \quad \text{as} \quad k\to\infty.$$

This gives the weak limit

$$\begin{aligned} & \text{weak-}\lim_k s_k B(z_{kj},z_{kj})^{1/2} B(s_k z_{kj}, s_k z_{kj})^{-1/2}(w_{kj}) \\ = & \sqrt{1-t_j^2}\, w_j + \left(1-t_j^2-\sqrt{1-t_j^2}\right)\langle w_j,\xi_j\rangle\,\xi_j. \end{aligned}$$

Hence

$$\begin{aligned} x_j &= \text{weak-}\lim_k \big(t_{kj}^2 z_{kj} + s_k B(z_{kj},z_{kj})^{1/2} B(s_k z_{kj}, s_k z_{kj})^{-1/2}(w_{kj})\big) \\ &= t_j^2\,\xi_j + \sqrt{1-t_j^2}\, w_j + \left(1-t_j^2-\sqrt{1-t_j^2}\right)\langle w_j,\xi_j\rangle\,\xi_j \\ &= t_j^2\,\xi_j + B(t_j\xi_j,t_j\xi_j)^{1/2}(w_j) \in t_j^2\,\xi_j + B(t_j\,\xi_j,t_j\,\xi_j)^{1/2}(\overline{D}_j) \end{aligned}$$

by (2.2.1).

Since $j\in J$ implies $\|\xi_j\|=1$ and, on the other hand, $j\notin J$ implies $t_j=0$, we have proved

$$S(\boldsymbol{\xi},\lambda)\subset S_1(\xi_1,\lambda)\times\cdots\times S_p(\xi_p,\lambda).$$

Conversely, given $\boldsymbol{u}=(u_1,\ldots,u_p)\in S_1(\xi_1,\lambda)\times\cdots\times S_p(\xi_p,\lambda)$, we have

$$u_j = t_j^2\,\xi_j + B(t_j\,\xi_j,t_j\,\xi_j)^{1/2}(v_j) \qquad (j\in\{1,\ldots,p\})$$

for some $v_j\in\overline{D}_j$, where $u_j=v_j$ if $t_j=0$.

For $k=1,2,\ldots$ and $j\in\{1,\ldots,p\}$, define $u_{kj}$ by

$$u_{kj} := \boldsymbol{c}_k(\lambda)_j + s_k B(z_{kj},z_{kj})^{1/2} B(s_k z_{kj}, s_k z_{kj})^{-1/2}(v_j).$$

Then $\boldsymbol{u}_k := (u_{k1},\ldots,u_{kp})\in D_k[\lambda]$. Since $u_j=\lim_k u_{kj}$, we have $\boldsymbol{u}=\lim_k \boldsymbol{u}_k\in S(\boldsymbol{\xi},\lambda)$. $\square$

*Remark* 3.2.9. We note that a description of invariant domains similar to our horospheres has been obtained with a different approach in [34] for polydiscs and compact holomorphic maps on Hilbert balls.

**Definition 3.2.10.** For the fixed-point-free holomorphic map $f : D \longrightarrow D$ in Proposition 3.2.8, we call the boundary point $\boldsymbol{\xi}$ a *Wolff point* of $f$.

We see from the above result that the closed horosphere $S(\boldsymbol{\xi}, \lambda)$ is convex. In fact, it is affinely homeomorphic to $\overline{D}$ since for each $j \in J$, the map

$$\varphi_j : x \in \overline{D}_j \mapsto t_j^2\, \xi_j + B(t_j\xi_j, t_j\xi_j)^{1/2}(x) \in \overline{D}_j \tag{3.2.6}$$

is an affine homeomorphism with image $S_j(\xi_j, \lambda)$. In particular, $S(\boldsymbol{\xi}, \lambda)$ is weakly compact. Moreover, we have $\varphi_j(D_j) \subset D_j$ since $\varphi_j$ is holomorphic and $\varphi_j(0) = t_j^2\xi_j \in D_j$ and the maximum modulus principle applies [11, Lemma 2]. It follows that $S_j(\xi_j, \lambda) = \varphi_j(\overline{D}_j)$ has interior $\varphi_j(D_j)$ and $S_j(\xi_j, \lambda) = \overline{\varphi_j(D_j)} = \overline{S_j(\xi_j, \lambda) \cap D_j}$. This gives

$$S(\boldsymbol{\xi}, \lambda) = \overline{S(\boldsymbol{\xi}, \lambda) \cap D}.$$

The closed horospheres $S(\boldsymbol{\xi}, \lambda)$ play an important role in the study of iterations of $f$ since for each $\boldsymbol{x} \in S(\boldsymbol{\xi}, \lambda) \cap D$ they contain the image $h(\boldsymbol{x})$ for each limit point $h$ of the iterates $(f^n)$. If $f$ is compact, then $h(D) \subset \partial D$ (*cf.* [31]) and hence it is useful to have a knowledge of the intersection $S(\boldsymbol{\xi}, \lambda) \cap \partial D$. We note from Proposition 3.2.8 that $S_j(\xi_j, \lambda) \cap \partial D_j = \partial D_j$ for $j \notin J$. However, for $j \in J$, the intersection is a singleton as shown below.

**Lemma 3.2.11.** *Let $f : D \longrightarrow D$ be the fixed-point-free holomorphic map in Proposition 3.2.8, with the Wolff point $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p) \in \partial D$ and the set $J \subset \{1, \dots, p\}$. For the closed horosphere $S(\boldsymbol{\xi}, \lambda) = \prod_{j=1}^{p} S_j(\xi_j, \lambda)$ at $\boldsymbol{\xi}$, we have*

*$S_j(\xi_j, \lambda) \cap \partial D_j = \{\xi_j\}$ for $j \in J$, and*

$$S(\boldsymbol{\xi}, \lambda) \cap \partial D = \bigcup_{j=1}^{p} S_1(\xi_1, \lambda) \times \cdots \times \Delta_j \times \cdots S_p(\xi_p, \lambda),$$

*where $\Delta_j = \{\xi_j\}$ for $j \in J$ and $\Delta_j = \partial D_j$ for $j \notin J$.*

*Proof.* Let $j \in J$ and pick $u_j \in S_j(\xi_j, \lambda) \cap \partial D_j$. We have

$$S_j(\xi_j, \lambda) = t_j^2 \xi_j + B(t_j \xi_j, t_j \xi_j)^{1/2}(\overline{D}_j),$$

where $t_j = \sqrt{\frac{\lambda}{\lambda + \gamma_j}} \in (0,1)$ and $\gamma_j = \liminf_{k\to\infty} \frac{1-\|z_{kj}\|^2}{1-\|\boldsymbol{z}_k\|^2} < \infty$. Choose any $u_i \in S_i(\xi_i, \lambda)$ for each $i \in \{1, \dots, p\} \backslash \{j\}$. Then $\boldsymbol{u} = (u_1, \dots, u_p) \in S(\boldsymbol{\xi}, \lambda) \cap \partial D$ and there exists a sequence $(\boldsymbol{x}_k)$ in $D_k[\lambda]$ such that $\boldsymbol{u} = \lim_k \boldsymbol{x}_k$, where $\boldsymbol{x}_k = (x_{k1}, \dots, x_{kp})$ and $u_j = \lim_k x_{kj} \in \partial D_j$. We have from (2.2.5) that

$$\frac{|1 - \langle x_{kj}, z_{kj} \rangle|^2}{1 - \|x_{kj}\|^2} = \frac{1 - \|z_{kj}\|^2}{1 - \|g_{-z_{kj}}(x_{kj})\|^2} \le \frac{1 - s_k^2 \|z_{kj}\|^2}{1 - s_k^2},$$

where

$$\frac{1 - s_k^2 \|z_{kj}\|^2}{1 - s_k^2} \to \frac{1}{t_j^2}$$

as $k \to \infty$. Hence

$$|1 - \langle x_{kj}, z_{kj} \rangle|^2 \le \left( \frac{1 - s_k^2 \|z_{kj}\|^2}{1 - s_k^2} \right) (1 - \|x_{kj}\|^2) \to 0 \quad \text{as } k \to \infty \tag{3.2.7}$$

and $|1 - \langle u_j, \xi_j \rangle|^2 = \lim_k |1 - \langle x_{kj}, z_{kj} \rangle|^2 = 0$. This gives $u_j = \xi_j$ and hence $S_j(\xi_j, \lambda) \cap \partial D_j = \{\xi_j\}$. This, together with a remark before the lemma, implies the last assertion. $\square$

*Remark* 3.2.12. The above proof also shows that each $u \in S_j(\xi_j, \lambda)$ with $j \in J$ satisfies $|1 - \langle u, \xi_j \rangle|^2 \le (1 - \|u\|^2)/t_j^2$. Also, since $J \neq \emptyset$, we have $S(\boldsymbol{\xi}, \lambda) \cap \partial D = \{\boldsymbol{\xi}\}$ for a single Hilbert ball $D = D_1$.

*Remark* 3.2.13. We note that, for every point $\boldsymbol{y} \in D$ one can find a $\mu > 0$ such that the closed horosphere $S(\boldsymbol{\xi}, \mu)$ contains $\boldsymbol{y}$. Indeed, define $r_k := \|g_{-\boldsymbol{z}_k}(\boldsymbol{y})\|$

and, by Lemma 3.2.2, we have $\lim_k r_k = 1$. Next define $\mu$ by

$$\mu := \lim_k \frac{1 - r_k^2}{1 - \|\boldsymbol{z}_k\|^2}.$$

We then use the sequence $(r_k)$ in place of the sequence $(s_k)$ in our previous construction and one can readily confirm $\boldsymbol{y} \in S(\boldsymbol{\xi}, \mu)$. Moreover, for a single Hilbert ball $D = D_1$, the point $\boldsymbol{y}$ is on the boundary of $S(\boldsymbol{\xi}, \mu)$, which we shall now show.

Our explicit description of the horosphere in this case is

$$S(\boldsymbol{\xi}, \mu) = t^2\boldsymbol{\xi} + B(t\boldsymbol{\xi}, t\boldsymbol{\xi})^{1/2}(\overline{D}),$$

where $t = \sqrt{\frac{\mu}{\mu+1}} = \sqrt{\frac{1-\|\boldsymbol{y}\|^2}{1-\|\boldsymbol{y}\|^2+|1-\langle \boldsymbol{y}, \boldsymbol{\xi}\rangle|^2}} \in (0, 1)$.

By our discussion immediately before Lemma 3.2.11, the interior of $S(\boldsymbol{\xi}, \mu)$ is $\varphi(D)$ where $\varphi : \boldsymbol{x} \in \overline{D} \mapsto t^2\boldsymbol{\xi} + B(t\boldsymbol{\xi}, t\boldsymbol{\xi})^{1/2}(\boldsymbol{x}) \in \overline{D}$. As $\boldsymbol{y} \in S(\boldsymbol{\xi}, \mu)$ there exists a $\boldsymbol{u} \in \overline{D}$ such that

$$\begin{aligned} \boldsymbol{y} &= t^2\boldsymbol{\xi} + B(t\boldsymbol{\xi}, t\boldsymbol{\xi})^{1/2}(\boldsymbol{u}) \\ &= t^2\boldsymbol{\xi} + \sqrt{1-t^2}\boldsymbol{u} + (1 - t^2 - \sqrt{1-t^2})\langle \boldsymbol{u}, \boldsymbol{\xi}\rangle\boldsymbol{\xi} \\ &= t^2\boldsymbol{\xi} + (1-t^2)\boldsymbol{u}_2 + \sqrt{1-t^2}\boldsymbol{u}_1, \end{aligned}$$

where $\boldsymbol{u}_k = P_k(\boldsymbol{\xi})(\boldsymbol{u})$ for $k = 1, 2$. Taking appropriate Peirce projections gives

$$\boldsymbol{u}_1 = \frac{\boldsymbol{y}_1}{\sqrt{1-t^2}} \quad \text{and} \quad \boldsymbol{u}_2 = \frac{\boldsymbol{y}_2 - t^2\boldsymbol{\xi}}{1-t^2},$$

where $\boldsymbol{y}_k = P_k(\boldsymbol{\xi})(\boldsymbol{y})$ for $k = 1, 2$.

We shall establish that $\boldsymbol{y}$ is on the boundary of $S(\boldsymbol{\xi}, \mu)$ by demonstrating that $\boldsymbol{u} \in \partial D$. A calculation gives

$$1 - \|\boldsymbol{y}\|^2 + |1 - \langle \boldsymbol{y}, \boldsymbol{\xi}\rangle|^2 + \|\boldsymbol{y}_1\|^2 \;=\; 2(1 - \mathrm{Re}\langle \boldsymbol{y}, \boldsymbol{\xi}\rangle).$$

Therefore

$$\begin{aligned}
2(1-t^2)(1-\mathrm{Re}\langle \boldsymbol{y},\boldsymbol{\xi}\rangle) &= \frac{1-\|\boldsymbol{y}\|^2+|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2+\|\boldsymbol{y}_1\|^2}{1-\|\boldsymbol{y}\|^2+|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2}|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2 \\
&= \left(1+\frac{\|\boldsymbol{y}_1\|^2}{1-\|\boldsymbol{y}\|^2+|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2}\right)|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2 \\
&= |1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2+(1-t^2)\|\boldsymbol{y}_1\|^2.
\end{aligned}$$

Using this formula we can calculate the squared norm of $\boldsymbol{u}_2$:

$$\begin{aligned}
\|\boldsymbol{u}_2\|^2 &= \left|\frac{\langle \boldsymbol{y},\boldsymbol{\xi}\rangle-t^2}{1-t^2}\right|^2 \\
&= \left|\frac{1-t^2-(1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle)}{1-t^2}\right|^2 \\
&= \left|1-\frac{1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle}{1-t^2}\right|^2 \\
&= 1-\frac{1}{(1-t^2)^2}\left(2(1-t^2)(1-\mathrm{Re}\langle \boldsymbol{y},\boldsymbol{\xi}\rangle)-|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2\right) \\
&= 1-\frac{1}{(1-t^2)^2}\left(|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2+(1-t^2)\|\boldsymbol{y}_1\|^2-|1-\langle \boldsymbol{y},\boldsymbol{\xi}\rangle|^2\right) \\
&= 1-\frac{\|\boldsymbol{y}_1\|^2}{1-t^2}.
\end{aligned}$$

Employing these calculations and the orthogonality of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in the Hilbert space $V=V_1$ we can calculate the squared norm of $\boldsymbol{u}$:

$$\begin{aligned}
\|\boldsymbol{u}\|^2 &= \|\boldsymbol{u}_1\|^2+\|\boldsymbol{u}_2\|^2 \\
&= \frac{\|\boldsymbol{y}_1\|^2}{1-t^2}+1-\frac{\|\boldsymbol{y}_1\|^2}{1-t^2} \\
&= 1.
\end{aligned}$$

## 3.3 Limit functions of iterates of holomorphic maps

We recall that the Denjoy-Wolff theorem for the complex disc $\mathbb{D}$ states that the iterates $(f^n)$ of a fixed-point-free holomorphic self-map $f$ on $\mathbb{D}$ converge in the compact-open topology to a constant map with unit-modulus value. This theorem has been extended to finite-dimensional Hilbert balls in [21, 33]. If $f$ is a *compact* holomorphic map on an infinite-dimensional Hilbert ball $D$, then every subsequence of the iterates $(f^n)$ admits a subsequence converging locally uniformly to a holomorphic map on $D$ [11]. This is a crucial ingredient in the proof of a Denjoy-Wolff theorem for $D$ in [11]. In contrast, it is not clear *a priori* that, without compactness, $(f^n)$ would still admit a subsequential limit. Indeed, it has been shown in [30, Example 3.1] that there exists a sequence of biholomorphic maps on the open unit ball of $\ell_2$, which has no convergent subsequence. Nevertheless, this obstacle can be circumvented by the existence of a single well-behaved orbit in $D$.

We first consider the more general case of a polyball $D = D_1 \times \cdots \times D_p$, with coordinate maps $\pi_j : (x_1, \ldots, x_p) \in \overline{D} \mapsto x_j \in \overline{D}_j$ for $j = 1, \ldots, p$. The following theorem generalises the iteration results in [2, 20] for polydiscs.

**Theorem 3.3.1.** *Let $f$ be a fixed-point-free holomorphic map on a polyball $D = D_1 \times \cdots \times D_p$, and compact if $p \geq 2$. Then there exist a boundary point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p) \in \partial D$ and a nonempty set $J \subset \{1, \ldots, p\}$ such that each limit function $h$ of the iterates $(f^n)$ satisfies $\xi_j \in \overline{\pi_j \circ h(D)}$ for all $j \in J$ and, $\pi_j \circ h(\cdot) = \xi_j$ whenever $\pi_j \circ h(D)$ meets $\partial D_j$.*

*Proof.* Let $h = \lim_k f^{n_k}$ be a limit function. By Proposition 3.2.5, there is a boundary point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p) \in \partial D$ such that for each $\lambda > 0$ the horosphere $S(\boldsymbol{\xi}, \lambda) \cap D$ is $f$-invariant.

By Proposition 3.2.8 and Lemma 3.2.11, there exists a nonempty subset

$J \subset \{1, \dots, p\}$ such that the closed horosphere $S(\boldsymbol{\xi}, \lambda) = \prod_{j=1}^{p} S_j(\xi_j, \lambda)$ satisfies $S_j(\xi_j, \lambda) \cap \partial D_j = \{\xi_j\}$ for $j \in J$, and $S_j(\xi_j, \lambda) = \overline{D}_j$ for $j \notin J$.

Fix $j \in J$. We show $\xi_j$ lies in the closure of $\pi_j \circ h(D)$. Choose a sequence $(\lambda_n)$ in $(0, \infty)$ which tends to $\infty$. For each $n$, there exists a $\boldsymbol{y}_n \in S(\boldsymbol{\xi}, \lambda_n) \cap D$ and so

$$h(\boldsymbol{y}_n) \in S(\boldsymbol{\xi}, \lambda_n)$$

where the $j$-th component of the closed horosphere $S(\boldsymbol{\xi}, \lambda_n)$ is of the form

$$S_j(\xi_j, \lambda_n) = t_{nj}^2 \xi_j + B(t_{nj}\xi_j, t_{nj}\xi_j)^{1/2}(\overline{D}_j)$$

where $t_{nj} = \sqrt{\frac{\lambda_n}{\lambda_n + \gamma_j}} \in (0, 1)$ and $\gamma_j = \liminf_{k \to \infty} \frac{1 - \|z_{kj}\|^2}{1 - \|\boldsymbol{z}_k\|^2} < \infty$. We note that $\lim_{n \to \infty} t_{nj} = 1$. Let $w_{nj} \in \overline{D}_j$ be such that

$$\pi_j \circ h(\boldsymbol{y}_n) = t_{nj}^2 \xi_j + B(t_{nj}\xi_j, t_{nj}\xi_j)^{1/2}(w_{nj}).$$

Since $\lim\limits_{n \to \infty} \|t_{nj}\xi_j\| = 1$, we have $\lim\limits_{n \to \infty} B(t_{nj}\xi_j, t_{nj}\xi_j)^{1/2}(w_{nj}) = 0$, as in the proof of Lemma 3.2.2. It follows that

$$\xi_j = \lim_n \pi_j \circ h(\boldsymbol{y}_n) \in \overline{\pi_j \circ h(D)}.$$

Finally, let $\pi_j \circ h(\boldsymbol{y}) \in \partial D_j$ for some $\boldsymbol{y} \in D$. Following Remark 3.2.13 there exists a $\mu_y > 0$ such that $\pi_j \circ h(\boldsymbol{y}) \in S_j(\xi_j, \mu_y) \cap \partial D_j = \{\xi_j\}$. It follows that $\pi_j \circ h(\boldsymbol{x}) = \xi_j$ for every $\boldsymbol{x} \in D$. Indeed, $\kappa(\pi_j \circ f^{n_k}(\boldsymbol{y}), \pi_j \circ f^{n_k}(\boldsymbol{x})) \leq \kappa(\boldsymbol{y}, \boldsymbol{x})$ implies $\pi_j \circ h(\boldsymbol{x}) \in \partial D_j$, by Lemma 3.2.2. Once again Remark 3.2.13 gives a $\mu_x > 0$ such that $\pi_j \circ h(\boldsymbol{x}) \in S_j(\xi_j, \mu_x) \cap \partial D_j = \{\xi_j\}$. $\square$

**Example 3.3.2.** The index set $J$ in Theorem 3.3.1 can be a proper subset of $\{1, \dots, p\}$ and the limit functions of $(f^n)$ need not be unique, even for the bidisc $\mathbb{D} \times \mathbb{D}$, as Hervé's work discussed in Section 3.1 shows. Indeed, one can construct a simple example. Let $a \in \mathbb{D} \backslash \{0\}$ and let $f$ be a self-map on $\mathbb{D} \times \mathbb{D}$

given by $f(z,w) = (g_a(z), iw)$, where $g_a$ is the Möbius transformation on $\mathbb{D}$. Then $(f^n)$ has four limit functions $h_k(z,w) = (a/|a|, i^k w)$ for $k = 1, \ldots, 4$ (see also Example 3.3.7).

For a single Hilbert ball $D$, the holomorphic map in Theorem 3.3.1 is not assumed to be compact and the Wolff point $\xi$ can lie in $\overline{h(D)} \backslash h(D)$ in which case the Denjoy-Wolff theorem fails. Indeed, there is an example in [39] of a fixed-point-free biholomorphic map $f$ on an infinite-dimensional Hilbert ball $D$ such that the iterates $(f^n)$ admit a limit function $h$ with $h(D) \subset D$. We now derive some criteria for $(f^n)$ to converge locally uniformly to $h$. We begin with a simple lemma.

**Lemma 3.3.3.** *Let $D$ be a Hilbert ball and $f : D \longrightarrow D$ be a holomorphic map such that the sequence $(f^n)$ converges pointwise to a constant map $h : D \longrightarrow \partial D$. Then $(f^n)$ converges locally uniformly to $h$.*

*Proof.* Let $h(D) = \{\xi\}$. To show locally uniform convergence, let $B \subset D$ be an open ball with $\operatorname{dist}(B, \partial D) > 0$ and let $D(\xi, \varepsilon)$ be an open ball of radius $\varepsilon > 0$ such that $\overline{D(\xi,\varepsilon)} \cap B = \emptyset$. We show that $f^n(B) \subset D(\xi, \varepsilon) \cap D$ from some $n$ onwards. This would complete the proof.

Suppose what we claim to show is false. Then we can find a subsequence $(f^{n_k})$ such that $f^{n_k}(b_k) \notin D(\xi, \varepsilon)$, where $b_k \in B$ and $f^{n_k}(b_k)$ converges weakly to some $v \in \overline{D}$, by weak compactness of $\overline{D}$.

Fix a point $y \in D$. We first note that $v \in \partial D$, for otherwise, Lemma 3.2.2 implies

$$\kappa(y, b_k) \geq \kappa(f^{n_k}(y), f^{n_k}(b_k)) = \tanh^{-1} \|g_{-f^{n_k}(y)}(f^{n_k}(b_k))\| \longrightarrow \infty$$

which contradicts the fact that

$$\sup_k\{\kappa(y, b_k)\} = \sup_k\{\tanh^{-1} \|g_{-y}(b_k)\|\} < \infty$$

since $b_k \in B$ for all $k$. Hence $\|v\| = 1$ and the sequence $(f^{n_k}(b_k))$ norm converges to $v$. Therefore $v \notin D(\xi, \varepsilon)$.

To complete the proof, compare the two sequences $(f^{n_k}(b_k))$ and $(f^{n_k}(y))$, having limits in the boundary $\partial D$. Since

$$\kappa(f^{n_k}(b_k), f^{n_k}(y)) \leq \kappa(b_k, y) \leq \sup_k \{\kappa(y, b_k)\} < \infty$$

we must have $v = \xi$ by Lemma 3.2.3, which contradicts $v \notin D(\xi, \varepsilon)$. $\square$

**Proposition 3.3.4.** *Let $f : D \longrightarrow D$ be a fixed-point-free holomorphic map on a Hilbert ball $D$ and let $a \in D$. Then either $\liminf_{n\to\infty} \|f^{2n}(a)\| < 1$ or $(f^n)$ converges locally uniformly to a constant map taking value at the boundary $\partial D$.*

*Proof.* Given $\liminf_n \|f^{2n}(a)\| \nless 1$, we must have $\lim_{n\to\infty} \|f^{2n}(a)\| = 1$. Let $\xi$ be the Wolff point of $f$ in Lemma 3.2.11.

We first show that $(f^{2n}(a))$ converges to $\xi$. By Remark 3.2.13 there exists a $\mu_a > 0$ such that $f^{2n}(a)$ is in the closed horosphere $S(\xi, \mu_a)$ and Remark 3.2.12 implies

$$|1 - \langle f^{2n}(a), \xi\rangle|^2 \leq \frac{1}{t_1^2}(1 - \|f^{2n}(a)\|^2) \longrightarrow 0 \quad \text{as } n \to \infty$$

which gives $\lim_n \langle f^{2n}(a), \xi\rangle = 1$ and $\lim_n f^{2n}(a) = \xi$.

We next show $\lim_n f^{2n+1}(a) = \xi$ . For this, it suffices to show that $\xi$ is the only weak limit point of the sequence $(f^{2n+1}(a))$. Let $(f^{2n_k+1}(a))$ be a subsequence of $(f^{2n+1}(a))$ weakly convergent to $\zeta \in \overline{D}$. If $\zeta \in D$, Lemma 3.2.2 implies

$$\kappa(a, f(a)) \geq \kappa(f^{2n_k}(a), f^{2n_k+1}(a)) = \tanh^{-1} \|g_{-f^{2n_k}(a)}(f^{2n_k+1}(a))\| \longrightarrow \infty$$

which is impossible. Hence we have $\zeta \in S(\xi, \mu_a) \cap \partial D = \{\xi\}$ and $\lim_n f^n(a) = \xi$.

Using Lemma 3.3.3, we complete the proof by showing that $(f^n)$ converges pointwise to the constant map $h(\cdot) = \xi$. Let $y \in D$. By Remark 3.2.13 there exists a $\mu_y > 0$ such that $y \in S(\xi, \mu_y)$. Let $v \in \overline{D}$ be any weak limit point of the sequence $(f^n(y))$. Then $(f^n(y))$ admits a subsequence $(f^{n_k}(y))$ weakly converging to $v$ and $v \in S(\xi, \mu_y)$. We show that $\|v\| = 1$. Otherwise, $v \in D$ implies

$$\begin{aligned} \kappa(a, y) &\geq \kappa(f^{n_k}(a), f^{n_k}(y)) \\ &= \tanh^{-1} \|g_{-f^{n_k}(a)}(f^{n_k}(y))\| \to \infty \end{aligned}$$

by Lemma 3.2.2, which is a contradiction. Hence we have $v \in S(\xi, \mu_y) \cap \partial D = \{\xi\}$. This shows that $\xi \in \partial D$ is the only weak limit point of the sequence $(f^n(y))$. Therefore $\lim_n f^n(y) = \xi$. As $y \in D$ was arbitrary, we have shown that $(f^n)$ converges pointwise to the constant map $h(\cdot) = \xi$. $\square$

If a fixed-point-free holomorphic map $f$ on $D$ has a convergent orbit, then its limit must lie in the boundary and the following corollary is immediate.

**Corollary 3.3.5.** *Let $f : D \longrightarrow D$ be a fixed-point-free holomorphic map on a Hilbert ball $D$. The following conditions are equivalent:*

*(i)* $\lim_{n\to\infty} \|f^{2n}(a)\| = 1$ *for some $a \in D$;*

*(ii) an orbit $(f^n(a))$ converges for some $a \in D$;*

*(iii) $(f^n)$ converges locally uniformly to a constant map taking value in $\partial D$.*

*Remark* 3.3.6. The Denjoy-Wolff theorem proved in [11, 31] for *compact* maps is a special case of the above result. Indeed, given a fixed-point-free holomorphic *compact* self-map $f$ on $D$ and $a \in D$, we have $\sup_k \|f^{n_k}(a)\| = 1$ for all subsequences $(f^{n_k})$ of $(f^n)$ (*cf.* [31, Theorem 3.1]). The example in [39] reveals that condition *(i)* above cannot be weakened to $\lim_k \|f^{n_k}(a)\| = 1$ for some subsequence $(f^{n_k})$. In fact, there is a biholomorphic map $f$ [39] on an infinite-dimensional Hilbert ball such that $\lim_k \|f^{n_k}(0)\| = 1$ for some subsequence $(f^{n_k})$ of $(f^n)$, but failing the Denjoy-Wolff theorem.

**Example 3.3.7.** Let $D$ be a Hilbert ball and $g_a$ the Möbius transformation induced by an element $a \in D \backslash \{0\}$. Then the iterates $(g_a^n(0))$ converge to $a/\|a\|$. Indeed, we have $g_a(0) = a$ and

$$g_a^2(0) = g_a(a) = a + \frac{B(a,a)^{1/2}(a)}{1 + \langle a, a \rangle} = \frac{2a}{1 + \|a\|^2}.$$

A simple computation gives

$$g_a^n(0) = \beta_n a$$

for some $\beta_n > 0$, where the sequence $(\beta_n)$ is increasing and bounded above by $1/\|a\|$, satisfying the recurrence relation

$$\beta_{n+1} = \frac{1 + \beta_n}{1 + \beta_n \|a\|^2}.$$

Hence $(\beta_n)$ converges to $1/\|a\|$. It follows that $(g_a^n)$ converges locally uniformly to the constant map with value $a/\|a\|$.

Remaining on a Hilbert ball $D$, next we consider the iteration of the biholomorphic map $\alpha g_a : D \longrightarrow D$ with $\alpha \in \partial \mathbb{D}$ and $a \in D \backslash \{0\}$, as an extension of the discussion in Section 1.1 of the dynamics of biholomorphic maps on the one-dimensional disc $\mathbb{D}$. In order to do so we first study its fixed points in the following lemmas.

**Lemma 3.3.8.** *Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $a \in D \backslash \{0\}$, where $D$ is a Hilbert ball. Then $z \in D$ is a fixed point of $\alpha g_a$ if and only if $z = \langle z, a \rangle a/\|a\|^2$ with*

$$\langle z, a \rangle^2 + (1 - \alpha)\langle z, a \rangle - \alpha \|a\|^2 = 0. \tag{3.3.1}$$

*Proof.* The Möbius transformation $g_a$ can be written, see (2.2.4), in the form

$$g_a(z) = \frac{a + P(z) + \sqrt{1 - \|a\|^2}\,(z - P(z))}{1 + \langle z, a \rangle}$$

where $P(\cdot) = \langle \cdot\,, a \rangle\, a/\|a\|^2$ denotes the orthogonal projection onto $\mathbb{C}a$, which is none other than the Peirce-2 projection $P_2(a/\|a\|)$.

It is clear $z \in D$ is a fixed point of $\alpha g_a$ if and only if

$$\begin{aligned} z &= \alpha g_a(z) \\ &= \frac{\alpha a + \alpha P(z) + \alpha\sqrt{1-\|a\|^2}\,(z - P(z))}{1 + \langle z, a\rangle}. \end{aligned}$$

By taking orthogonal projections, we deduce

$$\begin{aligned} P(z) &= \frac{\alpha a + \alpha P(z)}{1 + \langle z, a\rangle} \\ \iff \quad 0 &= \langle z, a\rangle^2 + (1-\alpha)\langle z, a\rangle - \alpha\|a\|^2 \end{aligned} \tag{3.3.2}$$

and

$$\begin{aligned} 0 &= \left(1 - \frac{\alpha\sqrt{1-\|a\|^2}}{1+\langle z, a\rangle}\right)(z - P(z)) \\ \iff \langle z, a\rangle &= -1 + \alpha\sqrt{1-\|a\|^2} \quad \text{or} \quad z = P(z). \end{aligned}$$

We complete the proof by showing that $\langle z, a\rangle = -1 + \alpha\sqrt{1-\|a\|^2}$ and (3.3.2) cannot hold simultaneously. Indeed, suppose $\langle z, a\rangle = -1 + \alpha\sqrt{1-\|a\|^2}$. The Cauchy–Schwarz inequality implies $\alpha \neq -1$ and substituting into (3.3.2) gives

$$\begin{aligned} 0 &= \langle z, a\rangle^2 + (1-\alpha)\langle z, a\rangle - \alpha\|a\|^2 \\ &= \left(-1 + \alpha\sqrt{1-\|a\|^2}\right)^2 + (1-\alpha)\left(-1 + \alpha\sqrt{1-\|a\|^2}\right) - \alpha\|a\|^2 \\ &= 1 - 2\alpha\sqrt{1-\|a\|^2} + \alpha^2(1-\|a\|^2) + (1-\alpha)\left(-1 + \alpha\sqrt{1-\|a\|^2}\right) - \alpha\|a\|^2 \\ &= 1 - 2\alpha\sqrt{1-\|a\|^2} + \alpha^2(1-\|a\|^2) \\ &\quad -1 + \alpha\sqrt{1-\|a\|^2} + \alpha - \alpha^2\sqrt{1-\|a\|^2} - \alpha\|a\|^2 \\ &= -\alpha(1+\alpha)\sqrt{1-\|a\|^2} + \alpha(1+\alpha)(1-\|a\|^2) \\ &= \alpha(1+\alpha)\sqrt{1-\|a\|^2}\left(\sqrt{1-\|a\|^2} - 1\right), \end{aligned}$$

which yields a contradiction. □

*Remark* 3.3.9. (3.3.1) is equivalent to

$$0 = \left\langle z, \frac{a}{\|a\|} \right\rangle^2 + \frac{(1-\alpha)}{\|a\|} \left\langle z, \frac{a}{\|a\|} \right\rangle - \alpha. \tag{3.3.3}$$

(3.3.3) is quadratic in $\langle z\,,\, a/\|a\|\rangle$, so we have two nonzero solutions in $\mathbb{C}$, counted according to multiplicity. By Lemma 3.3.8, a point $z_0 \in D$ is a fixed point of $\alpha g_a$ if and only if one $z_0 \in V_2(a/\|a\|)$ and one of the solutions of (3.3.3) is given by $\langle z_0\,,\, a/\|a\|\rangle$. Given a fixed point $z_0$, we have

$$\begin{aligned}
& \left\langle z_0, \frac{a}{\|a\|} \right\rangle^2 + \frac{(1-\alpha)}{\|a\|} \left\langle z_0, \frac{a}{\|a\|} \right\rangle - \alpha &=& \; 0 \\
\iff & \left(\overline{\left\langle z_0, \frac{a}{\|a\|} \right\rangle}\right)^2 + \frac{(1-\bar{\alpha})}{\|a\|} \overline{\left\langle z_0, \frac{a}{\|a\|} \right\rangle} - \bar{\alpha} &=& \; 0 \\
\iff & \left(\frac{1}{\overline{\left\langle z_0, \frac{a}{\|a\|} \right\rangle}}\right)^2 + \frac{(1-\alpha)}{\|a\|} \left(\frac{1}{\overline{\left\langle z_0, \frac{a}{\|a\|} \right\rangle}}\right) - \alpha &=& \; 0,
\end{aligned}$$

which means $1/\overline{\langle z_0\,,\, a/\|a\|\rangle}$ is also a solution of (3.3.3). As $\left|1/\overline{\langle z_0\,,\, a/\|a\|\rangle}\right| > 1$, this implies $z_0$ is the unique fixed point of $\alpha g_a$. Extending the domain of $\alpha g_a$ to $\overline{D}$, it is evident we either have exactly two fixed points, counted according to multiplicity, in the boundary $\partial D$ or exactly one fixed point inside $D$. We shall call fixed points in the boundary of $D$ *boundary fixed points.* There should then be no confusion in the statement that if $\alpha g_a$ is fixed-point-free, then it has at least one and at most two distinct boundary fixed points.

The following corollary is a direct consequence of the previous remark and Lemma 3.3.8 reworded so as to allow for $\alpha g_a$ extended to $\overline{D}$.

**Corollary 3.3.10.** *Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $a \in D\backslash\{0\}$, where $D$ is a Hilbert ball. If $\alpha g_a$ is fixed-point-free, then its boundary fixed points are given by*

$$\frac{\alpha - 1 \pm \sqrt{(\alpha-1)^2 + 4\alpha\|a\|^2}}{2\|a\|^2} a.$$

*Remark* 3.3.11. From Corollary 3.3.10, if $\alpha g_a$ is fixed-point-free, then it has exactly one boundary fixed point if and only if $(\alpha-1)^2+4\alpha\|a\|^2=0$, which is equivalent to $\alpha = 1-2\|a\|^2 \pm \left(2\|a\|\sqrt{1-\|a\|^2}\right) i$.

**Lemma 3.3.12.** *Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $a \in D\backslash\{0\}$, where $D$ is a Hilbert ball. Then $\alpha g_a$ is fixed-point-free if and only if $|1-\alpha| \le 2\|a\|$. In particular*

*(i)* $|1-\alpha| < 2\|a\|$ if and only if there exist exactly two distinct boundary fixed points of $\alpha g_a$ and no fixed points in $D$. In this case the boundary fixed points are $\frac{\alpha-1+\sqrt{(\alpha-1)^2+4\alpha\|a\|^2}}{2\|a\|^2}a$ and $\frac{\alpha-1-\sqrt{(\alpha-1)^2+4\alpha\|a\|^2}}{2\|a\|^2}a$.

*(ii)* $|1-\alpha| = 2\|a\|$ if and only if there exist exactly one boundary fixed point of $\alpha g_a$ and no fixed points in $D$. In this case we either have $\alpha = 1-2\|a\|^2 + \left(2\|a\|\sqrt{1-\|a\|^2}\right) i$ with boundary fixed point $\left(-1+\left(\sqrt{1/\|a\|^2-1}\right) i\right) a$ or $\alpha = 1-2\|a\|^2 - \left(2\|a\|\sqrt{1-\|a\|^2}\right) i$ with boundary fixed point $\left(-1-\left(\sqrt{1/\|a\|^2-1}\right) i\right) a$.

*(iii)* $|1-\alpha| > 2\|a\|$ if and only if there exist exactly one fixed point of $\alpha g_a$ in $D$ and no boundary fixed points.

*Proof.* By Lemma 3.3.8, $\alpha g_a$ is fixed-point-free if and only if

$$\frac{1}{2}\left|\alpha-1+\sqrt{(\alpha-1)^2+4\|a\|^2\alpha}\right| = \|a\| = \frac{1}{2}\left|\alpha-1-\sqrt{(\alpha-1)^2+4\|a\|^2\alpha}\right|$$

which, by Remark 3.3.9, holds if and only if

$$\left|\alpha-1+\sqrt{(\alpha-1)^2+4\|a\|^2\alpha}\right|^2 = \left|\alpha-1-\sqrt{(\alpha-1)^2+4\|a\|^2\alpha}\right|^2.$$

This is equivalent to

$$\begin{aligned}
0 &= 4\mathrm{Re}\left((1-\bar{\alpha})\sqrt{(1-\alpha)^2+4\|a\|^2\alpha}\right) \\
\iff 0 &= \mathrm{Re}\left(\sqrt{((1-\bar{\alpha})(1-\alpha))^2+4\|a\|^2\alpha(1-\bar{\alpha})^2}\right) \\
\iff 0 &= \mathrm{Re}\left(\sqrt{|1-\alpha|^4-4\|a\|^2|1-\alpha|^2}\right) \\
\iff 0 &= |1-\alpha|\mathrm{Re}\left(\sqrt{|1-\alpha|^2-4\|a\|^2}\right). \qquad (3.3.4)
\end{aligned}$$

Thus we recover the fact that the Möbius transformation $g_a$, which corresponds to $\alpha = 1$, is fixed-point-free and has the two boundary fixed points $\pm a/\|a\|$. If $\alpha \neq 1$, (3.3.4) is equivalent to:

$$0 = \operatorname{Re}\left(\sqrt{|1-\alpha|^2 - 4\|a\|^2}\right)$$

which amounts to the condition that $|1-\alpha| \leq 2\|a\|$.

In fact $|1-\alpha| = 2\|a\|$ if and only if there exist exactly one boundary fixed point and no fixed points in $D$. This can be seen from Remark 3.3.11 and the calculation

$$\begin{aligned} & \operatorname{Re}(\alpha) = 1 - 2\|a\|^2 \\ \Longleftrightarrow\ & \operatorname{Im}(\alpha) = \pm\sqrt{1-(1-2\|a\|^2)^2} = \pm 2\|a\|\sqrt{1-\|a\|^2} \\ \Longleftrightarrow\ & \alpha = 1 - 2\|a\|^2 \pm \left(2\|a\|\sqrt{1-\|a\|^2}\right) i, \end{aligned}$$

which shows we either have $\alpha = 1-2\|a\|^2+\left(2\|a\|\sqrt{1-\|a\|^2}\right) i$ with boundary fixed point $\left(-1+\left(\sqrt{1/\|a\|^2-1}\right) i\right) a$ or $\alpha = 1-2\|a\|^2-\left(2\|a\|\sqrt{1-\|a\|^2}\right) i$ with boundary fixed point $\left(-1-\left(\sqrt{1/\|a\|^2-1}\right) i\right) a$, by Corollary 3.3.10.

Therefore we have $|1-\alpha| < 2\|a\|$ if and only if there exist exactly two distinct boundary fixed points and no fixed points in $D$. In this case the boundary fixed points are $\frac{\alpha-1\pm\sqrt{(\alpha-1)^2+4\alpha\|a\|^2}}{2\|a\|^2}a$ from Corollary 3.3.10. □

**Example 3.3.13.** Extending Example 3.3.7, given $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $a \in D\backslash\{0\}$, the biholomorphic map $\alpha g_a$ is fixed-point-free in $D$ if and only if $|1-\alpha| \leq 2\|a\|$, by Lemma 3.3.12. In this case, the sequence $(\,(\alpha g_a)^n(0)\,)$ converges since $(\alpha g_a)^{n+1}(0) = (\alpha \mathrm{g}_{\|a\|})^n(\alpha\|a\|)a/\|a\|$, where

$$\mathrm{g}_{\|a\|} : \mathbb{D} \longrightarrow \mathbb{D}$$

denotes the Möbius transformation induced by $\|a\| \in \mathbb{D}$, and $\alpha \mathrm{g}_{\|a\|}$ is fixed-point-free. By Corollary 3.3.5, $((\alpha g_a)^n)$ converges locally uniformly to a constant map taking value in $\partial D$.

**Example 3.3.14.** Corollary 3.3.5 is false for a polyball $D = D_1 \times \cdots \times D_p$ with $p > 1$. Given $\boldsymbol{a} = (a_1, \ldots, a_p) \in D$ with Möbius transformation $g_{\boldsymbol{a}}$, the iterates $(g_{\boldsymbol{a}}^n)$ converge locally uniformly to a map $h : D \longrightarrow \overline{D}$

$$h(z_1, \ldots, z_k) = (w_1, \ldots, w_k)$$

which need not be constant, where

$$w_j = \begin{cases} a_j/\|a_j\| & \text{if } a_j \neq 0 \\ z_j & \text{if } a_j = 0. \end{cases}$$

*Remark* 3.3.15. Although our arguments in this section relate to holomorphic maps, the only feature of holomorphy relied upon was nonexpansiveness in the Kobayashi distance derived from the Schwarz-Pick Lemma.

Restarting the argument after Lemma 3.2.3, with the map $f$ taken now to be Kobayashi nonexpansive and not necessarily holomorphic, then one can still construct the sequence $(\boldsymbol{z}_k)$ which has a point $\boldsymbol{\xi} \in \partial D$ for norm limit. Indeed, by the arguments in [16, §4], one can show that for $0 < \alpha < 1$ the map $\alpha f$ is Lipschitz contractive, that is, there exists $\theta \in (0, 1)$ such that $\kappa(\alpha f(\boldsymbol{x}), \alpha f(\boldsymbol{y})) \leq \theta \kappa(\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in D$. It follows that $\alpha f$ has a fixed point in $\overline{\alpha f(D)} \subset D$. The construction of $(\boldsymbol{z}_k)$ and subsequent arguments then follow as before. In fact Proposition 3.2.5, Lemma 3.2.7, Proposition 3.2.8, Lemma 3.2.11, Theorem 3.3.1, Lemma 3.3.3, Proposition 3.3.4, Corollary 3.3.5 alongside necessary intermediary calculations and remarks all hold when recasted for Kobayashi nonexpansive self-maps.

# CHAPTER 4

## Holomorphic Dynamics on Rank-2 Domains

In this final chapter, we study holomorphic dynamics on irreducible rank-2 bounded symmetric domains which may be infinite-dimensional. These domains are the Lie balls and the open unit ball of the JBW*-triple $L(\mathbb{C}^2, H)$ for a Hilbert space $H$ of dimension at least 2. Together with the product of two Hilbert balls, they constitute the class of all infinite-dimensional rank-2 bounded symmetric domains. The dynamics on products of Hilbert balls has already been discussed in the previous chapter. The case of Lie balls has been studied in [10] and we will first review the results briefly. The new results for the case of the open unit ball of $L(\mathbb{C}^2, H)$, which are discussed in Sections 4.2, 4.3, 4.4 and 4.5, are our main focus in this chapter.

## 4.1 Lie balls

Let $V$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $^* : V \to V$ denote a *conjugation* - that is an involutive, conjugate linear isometry - satisfying $\langle x^*, y^* \rangle = \langle y, x \rangle$. Define a triple product $\{\cdot, \cdot, \cdot\} : V^3 \to V$ by

$$\{x, y, z\} \quad = \quad \frac{1}{2} \left( \langle x, y \rangle z + \langle z, y \rangle x - \langle x, z^* \rangle y^* \right)$$

for all $x, y, z \in V$ and the *spin factor norm* $\|\cdot\| : V \to [0, \infty)$ by

$$\|x\|^2 = \frac{1}{2}\left(\langle x, x\rangle + \sqrt{\langle x, x\rangle^2 - |\langle x, x^*\rangle|^2}\right).$$

The Banach space $E$ obtained from $V$ with the spin factor norm and above triple product is called a *spin factor*. The *Lie ball* $D$ is the open unit ball of $E$. More explicitly,

$$D = \left\{x \in E \,:\, \frac{1}{2}\left(\langle x, x\rangle + \sqrt{\langle x, x\rangle^2 - |\langle x, x^*\rangle|^2}\right) < 1\right\}.$$

The topological boundary of $D$ is denoted by $\partial D$.

The dynamics of a fixed-point-free holomorphic self-map $f$ on $D$, which is *compact*, meaning the norm closure $\overline{f(D)}$ is compact, were investigated in [10], to which we now turn. In order to examine the limit functions of $(f^n)$ it was first necessary to construct and then explicitly describe suitable $f$-invariant domains. First we note that using the Earle-Hamilton Fixed Point Theorem [16] it is possible to find a certain sequence $(z_k)$ in $D$ converging to a boundary point $\xi \in \partial D$. Then given a $\tau > 0$ there exists a sequence $(s_k)$ in $(0, 1)$ such that

$$\tau(1 - \|z_k\|^2) = 1 - s_k^2$$

from some $k$ onwards. As before, *cf.* Remark 3.2.6, we define the horosphere

$$S(\xi, \tau) = \left\{x \in \overline{D} \,:\, x = \lim_k x_k \text{ and } x_k \in g_{z_k}\left(\overline{D(0, s_k)}\right)\right\},$$

and denote its interior by $S_0(\xi, \tau)$. We note that $\xi \in S(\xi, \tau)$ and we have the invariance $f(S(\xi, \tau) \cap D) \subset S(\xi, \tau) \cap D$. The following theorem [10, Theorem 4.6] gives a useful description of the horospheres.

**Theorem 4.1.1.** *Let $f : D \to D$ be a fixed-point-free compact holomorphic map on a Lie ball $D$. Then there is a boundary point $\xi \in \partial D$ such that for each $\tau > 0$, there is a horosphere $S_0(\xi, \tau) \subset D$, which is a convex invariant domain* *of $f$ and satisfies $f(S(\xi,\tau)\cap D)\subset S(\xi,\tau)\cap D$ where $S(\xi,\tau)=\overline{S_0(\xi,\tau)}$.*

*Further, $\xi = e+\alpha e^*$ for some minimal tripotent $e\in\partial D$ with $|\alpha|\le 1$ and*

$$S_0(\xi,\tau)=\frac{\tau}{1+\tau}e+\frac{\alpha\sigma\tau}{1+\sigma\tau}e^*+B(\tau' e+\sigma' e^*,\tau' e+\sigma' e^*)^{1/2}(D)$$

*for some $\sigma\in[0,1]$, where $\tau'=\sqrt{\frac{\tau}{1+\tau}}$ and $\sigma'=\sqrt{\frac{\sigma\tau}{1+\sigma\tau}}$. If $|\alpha|<1$, we have*

$$S_0(\xi,\tau)=\frac{\tau}{1+\tau}e+B(\tau' e,\tau' e)^{1/2}(D).$$

This Jordan description of a horosphere was employed to produce the following analysis of the limit functions of $(f^n)$, [10, Theorem 6.17]. We refer the reader to the original paper for further details, including the definition of *non-degeneracy.*

**Theorem 4.1.2.** *Let $f: D\to D$ be a fixed-point-free compact holomorphic map on a Lie ball $D$ with boundary $\partial D$. Then there is a point $\xi=e+\alpha e^*\in\partial D$, where $e\in\partial D$ is a minimal tripotent and $|\alpha|\le 1$, such that each non-constant limit function $h$ of the iterates satisfies either of the following conditions.*

1. *$h(D)\subset e+\mathbb{D}e^*$.*
2. *$h(D)\subset u+\mathbb{D}u^*$ for some minimal tripotent $u$ and $(u+\overline{\mathbb{D}}u^*)\cap(e+\overline{\mathbb{D}}e^*)$ is a singleton in $e+\mathbb{T}e^*$.*

*If $\xi$ is non-degenerate, the singleton in (2) is $\{\xi\}$. If $h$ is a constant limit function, then it takes value in $e+\overline{\mathbb{D}}e^*$ and moreover, this value is $\xi$ if it is non-degenerate.*

## 4.2 Jordan geometry of $L(\mathbb{C}^2, H)$

In this section, we prepare the study of holomorphic iteration on the open unit ball of $L(\mathbb{C}^2, H)$. For this, we need to discuss the Jordan geometric structures of the JB*-triple $L(\mathbb{C}^2, H)$. We denote the topological boundary of the open unit ball $D$ of a Banach space $E$ by

$$\partial D = \{x \in E \,:\, \|x\| = 1\}.$$

Then the norm closure $\overline{D} = D \cup \partial D$ is the closed unit ball of $E$. Hereinafter, in Chapter 4, $V := L(\mathbb{C}^2, H)$ for short and $D$ denotes the open unit ball of $V$.

As noted in Section 2.2, $L(\mathbb{C}^2, H)$ is a JBW*-triple with predual given by

$$L(\mathbb{C}^2, H) \cong L(\mathbb{C}^2, H^{**}) \cong (\mathbb{C}^2 \hat{\otimes} H^*)^* \tag{4.2.1}$$

where $X \cong Y$ means $X$ is isometrically isomorphic to $Y$ and $\hat{\otimes}$ denotes the projective tensor product, *cf.* [15, p.230]. Since $L(\mathbb{C}^2, H)$ is a reflexive Banach space, the weak topology of $L(\mathbb{C}^2, H)$ coincides with the weak* topology.

In what follows, for each $y \in H$, the functional $f_y \in H^*$ is defined by $f_y(\cdot) = \langle \cdot, y \rangle$. By (4.2.1), each $T \in L(\mathbb{C}^2, H)$ identifies with an element $\tilde{T} \in (\mathbb{C}^2 \hat{\otimes} H^*)^*$ such that $\tilde{T}(z \otimes f_y) = \langle T(z), y \rangle$ for every $z \in \mathbb{C}^2$ and $y \in H$. Therefore we have the following criteria for weak convergence.

**Lemma 4.2.1.** *Let $(T_k)$ be a sequence in $L(\mathbb{C}^2, H)$ and $T \in L(\mathbb{C}^2, H)$. Then $(T_k)$ converges to $T$ weakly if and only if $\langle T_k(z), h \rangle \to \langle T(z), h \rangle$ for all $z \in \mathbb{C}^2$ and $h \in H$, which is equivalent to $(T_k(z))$ weakly converging to $T(z)$ in $H$ for all $z \in \mathbb{C}^2$.*

Where convenient we will write $T_k \xrightarrow{w} T$ or $T = \text{weak-}\lim_k T_k$ to mean that $(T_k)$ weakly converges to $T$ in $L(\mathbb{C}^2, H)$.

**Lemma 4.2.2.** *A nonzero tripotent in $L(\mathbb{C}^2, H)$ is either a minimal tripotent or a maximal tripotent, which is a sum of two triple orthogonal minimal tripotents.*

*Proof.* Let $e$ be a nonzero tripotent in $V$ with spectral decomposition (*q.v.*

Corollary 2.2.4)

$$e = \lambda u + \delta w,$$

where $u$ and $w$ are triple orthogonal minimal tripotents and $\lambda \geq \delta \geq 0$. As $e$ is nonzero, $1 = \|e\| = \max(\lambda, \delta) = \lambda$ implies $e = u + \delta w$. Now, as $e$ is a tripotent, we must have

$$\begin{aligned} u + \delta w &= \{u + \delta w, u + \delta w, u + \delta w\} \\ &= \{u, u, u\} + \delta^3\{w, w, w\} \\ &= u + \delta^3 w, \end{aligned}$$

which implies $(1-\delta)(1+\delta)\delta w = 0$, and so either $\delta w = 0$, in which case $e = u$ is a minimal tripotent, or $\delta = 1$, in which case $e = u + w$ is a maximal tripotent. □

We recall from Section 2.2 that each minimal tripotent in $L(\mathbb{C}^2, H)$ is a rank-one operator $a \otimes b$, with $a \in \mathbb{C}^2$, $b \in H$ and $\|a\| = \|b\| = 1$.

**Lemma 4.2.3.** *Let $e = a_e \otimes b_e$ be a minimal tripotent in $V$. The Peirce 0-space of $e$ is given by*

$$V_0(e) = \{x \in L(\mathbb{C}^2, H) : x(a_e) = 0 \textit{ and } \langle x(z), b_e \rangle = 0 \textit{ for all } z \in \mathbb{C}^2\}.$$

*Proof.* The follows immediately from the Peirce 0-projection,

$$\begin{aligned} P_0(e)(x) &= (I_H - ee^*) \circ x \circ (I_{\mathbb{C}^2} - e^* e) \\ &= (I_H - P_2(b_e)) \circ x \circ (I_{\mathbb{C}^2} - P_2(a_e)) \\ &= P_1(b_e) \circ x \circ P_1(a_e) \end{aligned}$$

for all $x \in V$, *q.v.* (2.2.8). □

**Corollary 4.2.4.** *Let $e = a_e \otimes b_e$ be a minimal tripotent in $V$. Then there exists an orthonormal basis $\{a_e, \nu\}$ of $\mathbb{C}^2$ such that the Peirce 0-space of $e$ is given by*

$$V_0(e) \;=\; \{x \in L(\mathbb{C}^2, H) \,:\, x(a_e) = 0 \text{ and } \langle x(\nu), b_e\rangle = 0\}.$$

*Proof.* Extending $a_e$ to an orthonormal basis $\{a_e, v\}$ of $\mathbb{C}^2$. We can write each $z \in \mathbb{C}^2$ as $z = \langle z, a_e\rangle a_e + \langle z, \nu\rangle \nu$. Let $x \in L(\mathbb{C}^2, H)$ with $x(a_e) = 0$. The equivalences

$$\begin{aligned}
&0 = \langle x(z), b_e\rangle \text{ for all } z \in \mathbb{C}^2\\
\Longleftrightarrow\ &0 = \langle z, a_e\rangle\langle x(a_e), b_e\rangle + \langle z, \nu\rangle\langle x(\nu), b_e\rangle \text{ for all } z \in \mathbb{C}^2\\
\Longleftrightarrow\ &0 = \langle x(\nu), b_e\rangle
\end{aligned}$$

yield the result.

□

*Remark* 4.2.5. From Corollary 4.2.4 we see that, given two triple orthogonal minimal tripotents $e = a_e \otimes b_e$ and $v = a_v \otimes b_v$ in $V$, we have

$$V_0(e) \;=\; \{x \in L(\mathbb{C}^2, H) \,:\, x(a_e) = 0 \text{ and } \langle x(a_v), b_e\rangle = 0\}.$$

We show below that every nonzero element in $V_0(e)$ is of rank-one.

**Lemma 4.2.6.** *Let $x \in V_0(e)$ and $x \neq 0$ where $e$ is a minimal tripotent in $V$. Then there exists a minimal tripotent $u \in V$ such that $x \in \mathbb{C}u$ and $u$ is triple orthogonal to $e$.*

*Proof.* As $x \in V_0(e)$, by Corollary 4.2.4 there exists an orthonormal basis $\{a_e, \nu\}$ of $\mathbb{C}^2$ such that $x(a_e) = 0$ and $\langle x(\nu), b_e\rangle = 0$.

As $x(z) = \langle z, a_e\rangle x(a_e) + \langle z, \nu\rangle x(\nu) = \langle z, \nu\rangle x(\nu)$ for all $z \in \mathbb{C}^2$ and $x \neq 0$, we have $x(\nu) \neq 0$.

Define the rank-one operator $u : \mathbb{C}^2 \to H$ by

$$u := \nu \otimes \frac{x(\nu)}{\|x(\nu)\|}.$$

The calculation

$$\begin{aligned}
\{u, u, u\} &= uu^*u \\
&= \left(\nu \otimes \frac{x(\nu)}{\|x(\nu)\|}\right) \circ \left(\frac{x(\nu)}{\|x(\nu)\|} \otimes \nu\right) \circ \left(\nu \otimes \frac{x(\nu)}{\|x(\nu)\|}\right) \\
&= \left\langle \left\langle \langle \cdot, \nu \rangle \frac{x(\nu)}{\|x(\nu)\|}, \frac{x(\nu)}{\|x(\nu)\|} \right\rangle \nu, \nu \right\rangle \frac{x(\nu)}{\|x(\nu)\|} \\
&= \nu \otimes \frac{x(\nu)}{\|x(\nu)\|} \\
&= u
\end{aligned}$$

shows that $u$ is a tripotent. Moreover, $u$ is minimal. Confirming $u$ is triple orthogonal to $e$ is routine, see Lemma 2.2.2. We now show $x \in \mathbb{C}u$. Indeed

$$\begin{aligned}
x(z) &= \langle z, \nu \rangle x(\nu) \\
&= \|x(\nu)\| \langle z, \nu \rangle \frac{x(\nu)}{\|x(\nu)\|} \\
&= \|x(\nu)\| u(z).
\end{aligned}$$

□

**Corollary 4.2.7.** *Given a minimal tripotent $e \in V$, the intersection $V_0(e) \cap \partial D$ consists of minimal tripotents.*

*Proof.* Let $x \in V_0(e) \cap \partial D$. Then by Lemma 4.2.6 there exists a minimal tripotent $x' \in V_0(e)$ such that $x = \alpha x'$ for some $\alpha \in \mathbb{C}$. The result follows, as $1 = \|x\| = \|\alpha x'\| = |\alpha|$.

□

We will denote by $D_0(e)$ the open unit ball $D \cap V_0(e)$ of the Peirce-0 space $V_0(e)$ with respect to the tripotent $e$.

**Lemma 4.2.8.** *Given a minimal tripotent $e \in V$, we have $V_0(e) \cap \partial D = \overline{D_0(e)} \backslash D_0(e)$*

*Proof.* First we show $V_0(e) \cap \partial D \subset \overline{D_0(e)} \backslash D_0(e)$. Let $x \in V_0(e) \cap \partial D$ then $\|x\| = 1$ and hence $x \in \overline{D_0(e)} \backslash D_0(e)$.

Next we show that $\overline{D_0(e)} \backslash D_0(e) \subset V_0(e) \cap \partial D$. Let $x \in \overline{D_0(e)} \backslash D_0(e)$, then there exists a sequence $(x_k)$ in $D_0(e)$ converging to $x$. Hence $x \in V_0(e)$, as $V_0(e)$ is norm closed. Now $1 \leq \|x\| = \lim \|x_k\| \leq 1$ implies $x \in V_0(e) \cap \partial D$. □

*Remark* 4.2.9. From Corollary 4.2.7 and Lemma 4.2.8, we deduce that each element in $\overline{D_0(e)}$ is of the form $\lambda v$ for some minimal tripotent $v$ and $\lambda \in \overline{\mathbb{D}}$.

As in Section 2.1, let $K_a$ denote the boundary component containing $a \in V$ and, recalling Proposition 2.1.18, given a tripotent $e \in V$, we have $K_e = e + D_0(e)$.

**Lemma 4.2.10.** *Let $e \in V$ be a nonzero tripotent and $K_e$ be the boundary component in $\partial D$ containing $e$. Then precisely one of the following holds.*

*(i) $e$ is maximal and $\overline{K_e} \backslash K_e$ is empty, or*

*(ii) $e$ is minimal and $\overline{K_e} \backslash K_e$ is the following set of maximal tripotents:*

$$\{e + v \,:\, v \text{ is a minimal tripotent triple orthogonal to } e\}.$$

*Proof.* Suppose $e$ is maximal. Then $K_e = \{e\}$ and *(i)* is immediate.

Now suppose $e$ is minimal. Then, by Lemma 4.2.8, we have $\overline{K_e} \backslash K_e = \overline{(e + D_0(e))} \backslash (e + D_0(e)) = e + \overline{D_0(e)} \backslash D_0(e) = e + V_0(e) \cap \partial D$, and the result follows from Corollary 4.2.7. □

**Lemma 4.2.11.** *Let $v_1 = a_{v_1} \otimes b_{v_1}$ and $v_2 = a_{v_2} \otimes b_{v_2}$ be two minimal tripotents in $V$. Then $v_1 = v_2$ if and only if $\langle a_{v_1}, a_{v_2} \rangle = \langle b_{v_1}, b_{v_2} \rangle \in \partial \mathbb{D}$.*

*Proof.* We note that $\|a_{v_1}\| = \|a_{v_2}\| = 1$ and $b_{v_1}$ and $b_{v_2}$ are both minimal and maximal tripotents in the Hilbert space $H$. If $v_1 = v_2$ then $a_{v_1} = \lambda a_{v_2}$

and $b_{v_1} = \lambda b_{v_2}$ for some $\lambda \in \partial\mathbb{D}$. Taking appropriate inner products gives $\lambda = \langle a_{v_1}, a_{v_2}\rangle = \langle b_{v_1}, b_{v_2}\rangle$.

On the other hand, if $\langle a_{v_1}, a_{v_2}\rangle = \langle b_{v_1}, b_{v_2}\rangle \in \partial\mathbb{D}$, we have $a_{v_1} = \langle a_{v_1}, a_{v_2}\rangle a_{v_2}$ and, since $P_0(b_{v_2}) = 0$, the calculation

$$\begin{aligned}
1 = \|b_{v_1}\|^2 &= \|P_2(b_{v_2})b_{v_1} + P_1(b_{v_2})b_{v_1}\|^2 \\
&= \|P_2(b_{v_2})b_{v_1}\|^2 + \|P_1(b_{v_2})b_{v_1}\|^2 \\
&= |\langle b_{v_1}, b_{v_2}\rangle|^2\|b_{v_2}\|^2 + \|P_1(b_{v_2})b_{v_1}\|^2 \\
&= 1 + \|P_1(b_{v_2})b_{v_1}\|^2
\end{aligned}$$

implies $b_{v_1} = P_2(b_{v_2})b_{v_1} = \langle b_{v_1}, b_{v_2}\rangle b_{v_2}$. Therefore

$$\begin{aligned}
v_1 &= a_{v_1} \otimes b_{v_1} \\
&= (\langle a_{v_1}, a_{v_2}\rangle a_{v_2}) \otimes (\langle b_{v_1}, b_{v_2}\rangle b_{v_2}) \\
&= \overline{\langle a_{v_1}, a_{v_2}\rangle}\langle b_{v_1}, b_{v_2}\rangle\,(a_{v_2} \otimes b_{v_2}) \\
&= \overline{\langle a_{v_1}, a_{v_2}\rangle}\langle a_{v_1}, a_{v_2}\rangle\,(a_{v_2} \otimes b_{v_2}) \\
&= v_2.
\end{aligned}$$

□

**Lemma 4.2.12.** *The Peirce-0 space of $V$ with respect to a minimal tripotent is strictly convex.*

*Proof.* Let $v_1$ and $v_2$ be two distinct norm-one points in the Peirce-0 space $V_0(e)$ of a minimal tripotent $e = a_e \otimes b_e \in V$. By Corollary 4.2.7, $v_1$ and $v_2$ are minimal tripotents triple orthogonal to $e$. Therefore $v_1$ and $v_2$ take the form $v_1 = a_{v_1} \otimes b_{v_1}$ and $v_2 = a_{v_2} \otimes b_{v_2}$, where $\{a_{v_1}, a_e\}$ and $\{a_{v_2}, a_e\}$ are both orthonormal bases of $\mathbb{C}^2$. As $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\| = 2$, the proof will be complete if $\|v_1 + v_2\| \neq 2$.

We have,

$$
\begin{aligned}
\|v_1 + v_2\|^2 &= \|a_{v_1} \otimes b_{v_1} + a_{v_2} \otimes b_{v_2}\|^2 \\
&= \sup_{z \in \mathbb{C}^2, \|z\|=1} \|a_{v_1} \otimes b_{v_1}(z) + a_{v_2} \otimes b_{v_2}(z)\|_H^2 \\
&= \sup_{z \in \mathbb{C}^2, \|z\|=1} \|a_{v_1} \otimes b_{v_1}(\langle z, a_{v_1}\rangle a_{v_1} + \langle z, a_e\rangle a_e) \\
&\qquad\qquad + a_{v_2} \otimes b_{v_2}(\langle z, a_{v_1}\rangle a_{v_1} + \langle z, a_e\rangle a_e)\|_H^2 \\
&= \sup_{z \in \mathbb{C}^2, \|z\|=1} \|a_{v_1} \otimes b_{v_1}(\langle z, a_{v_1}\rangle a_{v_1}) + a_{v_2} \otimes b_{v_2}(\langle z, a_{v_1}\rangle a_{v_1})\|_H^2 \\
&= \|a_{v_1} \otimes b_{v_1}(a_{v_1}) + a_{v_2} \otimes b_{v_2}(a_{v_1})\|_H^2 \sup_{z \in \mathbb{C}^2, \|z\|=1} |\langle z, a_{v_1}\rangle|^2 \\
&= \|b_{v_1} + \langle a_{v_1}, a_{v_2}\rangle b_{v_2}\|_H^2 \\
&= 1 + |\langle a_{v_1}, a_{v_2}\rangle|^2 + 2\mathrm{Re}\langle a_{v_1}, a_{v_2}\rangle\langle b_{v_2}, b_{v_1}\rangle \\
&\leq 2\left(1 + \mathrm{Re}\langle a_{v_1}, a_{v_2}\rangle\langle b_{v_2}, b_{v_1}\rangle\right).
\end{aligned}
$$

As $v_1 \neq v_2$, we have $\langle a_{v_1}, a_{v_2}\rangle\langle b_{v_2}, b_{v_1}\rangle \neq 1$, by Lemma 4.2.11, and therefore $\mathrm{Re}\langle a_{v_1}, a_{v_2}\rangle\langle b_{v_2}, b_{v_1}\rangle < 1$, which gives $\|v_1 + v_2\| < 2$. □

**Lemma 4.2.13.** *Let $A$ and $B$ be two boundary components of $D$ contained in $\partial D$. If the intersection $\overline{A} \cap \overline{B}$ contains two distinct elements, then $A = B$.*

*Proof.* There exist nonzero tripotents $e$ and $u$ such that $A = K_e$ and $B = K_u$. If the intersection $\overline{A} \cap \overline{B} = \overline{K_e} \cap \overline{K_u}$ contains two distinct elements, then $e$ and $u$ must be minimal tripotents. Let $x_1, x_2 \in \overline{K_e} \cap \overline{K_u}$ such that $x_1 \neq x_2$. We have

$$x_i = e + \lambda_i v_i = u + \sigma_i w_i \quad (i = 1, 2)$$

where $v_1$ and $v_2$ (resp. $w_1$ and $w_2$) are minimal tripotents triple orthogonal to $e$ (resp. $u$), and $\lambda_i, \sigma_i \in \overline{\mathbb{D}}$ for $i = 1, 2$. One can readily see that

$$x_i \in K_e \iff |\lambda_i| < 1 \iff |\sigma_i| < 1 \iff x_i \in K_u \quad (i = 1, 2),$$

where the middle equivalence comes from the fact that if either $|\lambda_i|$ or $|\sigma_i|$ is 1 then $x_i$ is a maximal tripotent, as the sum of two minimal triple orthogonal tripotents, and therefore the other of $|\lambda_i|$ or $|\sigma_i|$ must be 1 as well. Therefore, if either $|\lambda_i|$ or $|\sigma_i|$ is strictly less than one, then we have $K_e = K_u$. Hence, we may assume, without loss of generality, that

$$x_i = e + \lambda_i v_i = u + \sigma_i w_i \quad (i = 1, 2),$$

where $|\lambda_i| = 1 = |\sigma_i|$ for $i = 1, 2$.

By Lemma 4.2.12, we know that both Peirce-0 spaces $V_0(e)$ and $V_0(u)$ are strictly convex and therefore

$$\left\| \frac{1}{2}(\lambda_1 v_1 + \lambda_2 v_2) \right\| < 1 \text{ and } \left\| \frac{1}{2}(\sigma_1 w_1 + \sigma_2 w_2) \right\| < 1,$$

as $\lambda_1 v_1$ and $\lambda_2 v_2$ are distinct elements of $V_0(e) \cap \partial D$, and $\sigma_1 w_1$ and $\sigma_2 w_2$ are distinct element of $V_0(u) \cap \partial D$. Using this in the following calculation

$$\frac{1}{2}(x_1 + x_2) = e + \frac{1}{2}(\lambda_1 v_1 + \lambda_2 v_2) = u + \frac{1}{2}(\sigma_1 w_1 + \sigma_2 w_2) \in K_e \cap K_u$$

implies $K_e = K_u$, as distinct boundary components are disjoint. $\square$

Next, we give an example showing that the intersection $\overline{K_e} \cap \overline{K_u}$ can be nonempty even when $K_e \neq K_u$ and the minimal tripotents $e$ and $u$ are not triple orthogonal.

**Example 4.2.14.** Consider the infinite-dimensional Cartan factor $L(\mathbb{C}^2, \ell_2)$, where $\ell_2$ is the usual Hilbert space of square-summable sequences of complex numbers. Consider the minimal tripotents

$$e = a_e \otimes b_e \qquad \text{and} \qquad u = a_u \otimes b_u$$

$$v = a_v \otimes b_v \qquad\qquad w = a_w \otimes b_w$$

defined by

$$
\begin{aligned}
a_e &= (1,0) \quad \text{and} \quad b_e = (1,0,0,0,\dots),\\
a_v &= (0,1) \quad \text{and} \quad b_v = (0,1,0,0,\dots),\\
a_u &= (1,1)/\sqrt{2} \quad \text{and} \quad b_u = (1,1,0,0,\dots)/\sqrt{2},\\
a_w &= (1,-1)/\sqrt{2} \quad \text{and} \quad b_w = (1,-1,0,0,\dots)/\sqrt{2}.
\end{aligned}
$$

It is easily verified that $e$ and $v$ are triple orthogonal, as are $u$ and $w$. Note that, although $e$ and $u$ are not triple orthogonal, $e + v = u + w$ is a maximal tripotent which belongs to $\overline{K_e} \cap \overline{K_u}$.

## 4.3 Wolff point

Let $D$ be the open unit ball of $L(\mathbb{C}^2, H)$ and $f : D \to D$ be a fixed-point-free holomorphic map which is *compact*, that is $\overline{f(D)}$ is compact. Take any strictly increasing sequence $(\gamma_k)$ in the interval $(0,1)$ converging to 1. Define $f_k := \gamma_k f$ for all $k \in \mathbb{N}$. By the Earle-Hamilton Fixed Point Theorem [16], each $f_k$ has a unique fixed point $z_k$ in $D$. The sequence $(\gamma_k^{-1} z_k)$ is contained in the norm compact set $\overline{f(D)}$ and, by choosing a subsequence, assume that $(\gamma_k^{-1} z_k)$ norm converges to $\xi \in \overline{D}$. It is easy to see that $(z_k)$ converges to the same limit $\xi$. Note that $\xi \in \partial D$ for otherwise, $f(\xi) = \lim_k \gamma_k^{-1} f_k(z_k) = \lim_k \gamma_k^{-1} z_k = \xi$, which contradicts $f$ being fixed-point-free. We will call $\xi$ a *Wolff point* of $f$ which, as in the case of other symmetric domains discussed before, plays an important role in the dynamics of $f$. We study the spectral representation of $\xi$ in this section for later applications.

Choose a spectral decomposition of each $z_k$:

$$
z_k = \alpha_k e_k + \beta_k v_k,
$$

where $1 > \alpha_k \geq \beta_k \geq 0$ and $e_k$ and $v_k$ are triple orthogonal minimal tripotents.

We have $\alpha_k = \|z_k\| \to 1$. Passing to a subsequence if necessary, we may assume that $(\beta_k)$ converges to $\beta$.

Let

$$e_k = a_{e_k} \otimes b_{e_k} \text{ and } v_k = a_{v_k} \otimes b_{v_k},$$

with $a_{e_k}, a_{v_k} \in \mathbb{C}^2$ and $b_{e_k}, b_{v_k} \in H$, all of unit norm. By Lemma 2.2.2, $a_{e_k} \perp a_{v_k}$ and $b_{e_k} \perp b_{v_k}$ in their respective Hilbert spaces. Now $(a_{e_k})$ and $(a_{v_k})$ are bounded sequences in $\mathbb{C}^2$ and, passing to subsequences, if necessary, we may assume they have norm limits $a_e$ and $a_v$ respectively. By continuity of the inner product, $a_e$ and $a_v$ are orthogonal in $\mathbb{C}^2$. Similarly, by weak compactness of the closed unit ball in $H$ and choosing subsequences if necessary, we may assume $(b_{e_k})$ and $(b_{v_k})$ converge weakly to $b_e$ and $b_v$ respectively.

Let $e = a_e \otimes b_e$ and $v = a_v \otimes b_v$. Then we have $e_k \xrightarrow{w} e$ and $v_k \xrightarrow{w} v$, which imply $e_k^*(h) \to e(h)$ and $v_k^*(h) \to v(h)$ in $\mathbb{C}^2$ for each $h \in H$.

As $(z_k)$ norm converges to $\xi$, we must have

$$\xi \quad = \quad e + \beta v.$$

We are going to show that $e$ is a tripotent and, if $\beta \neq 0$, so is $v$. We retain the previous notation in the following lemmas.

**Lemma 4.3.1.** *We have* $\{e_k, u, e_k\} \xrightarrow{w} \{e, u, e\}$ *and* $\{v_k, u, v_k\} \xrightarrow{w} \{v, u, v\}$ *for all* $u \in V$.

*Proof.* We show $\{e_k, u, e_k\} \xrightarrow{w} \{e, u, e\}$. The proof of the other weak limit is similar. We have

$$\langle e_k u^* e_k(\mu), h\rangle_H \quad = \quad \langle u^* e_k(\mu), e_k^* h\rangle_{\mathbb{C}^2},$$

where $u^* : H \to \mathbb{C}^2$ is norm continuous and hence is weak-weak continuous, *q.v.* for example [14, Theorem 5].

Therefore, as $e_k(\mu) \xrightarrow{w} e(\mu)$ in $H$, we have

$$u^*(e_k(\mu)) \quad \xrightarrow{w} \quad u^*(e(\mu))$$

in $\mathbb{C}^2$, and hence the norm convergence $u^*(e_k(\mu)) \to u^*(e(\mu))$ in $\mathbb{C}^2$.

It follows that

$$\begin{aligned}
\langle e_k u^* e_k(\mu), h\rangle_H &= \langle u^* e_k(\mu), e_k^* h\rangle_{\mathbb{C}^2} \\
&\to \langle u^* e(\mu), e^* h\rangle_{\mathbb{C}^2} \\
&= \langle e u^* e(\mu), h\rangle_H
\end{aligned}$$

that is, $e_k u^* e_k \xrightarrow{w} e u^* e$. □

**Lemma 4.3.2.** *Let $(w_k)$ be a sequence in $V$ norm converging to $0$. Then we have*

$$\{e_k, w_k, e_k\} \quad \xrightarrow{w} \quad 0 \tag{4.3.1}$$

*and*

$$\{v_k, w_k, v_k\} \quad \xrightarrow{w} \quad 0. \tag{4.3.2}$$

*Proof.* We prove (4.3.1). The proof of (4.3.2) is similar. It is sufficient to note that

$$\begin{aligned}
|\langle e_k w_k^* e_k(\mu), h\rangle_H| &= |\langle w_k^* e_k(\mu), e_k^* h\rangle_{\mathbb{C}^2}| \\
&\leq \|w_k^* e_k(\mu)\|_{\mathbb{C}^2} . \|e_k^* h\|_{\mathbb{C}^2} \\
&\leq \|w_k^*\|_{L(H,\mathbb{C}^2)} . \|e_k(\mu)\|_H . \|e_k^*\|_{L(H,\mathbb{C}^2)} . \|h\|_H \\
&\leq \|w_k\|_V . \|\mu\|_{\mathbb{C}^2} . \|h\|_H \\
&\to 0
\end{aligned}$$

for all $\mu \in \mathbb{C}^2$ and $h \in H$. □

**Corollary 4.3.3.** *We have* $\{e_k, z_k, e_k\} \xrightarrow{w} \{e, \xi, e\}$ *and* $\{v_k, z_k, v_k\} \xrightarrow{w} \{v, \xi, v\}$.

*Proof.* Since the sequence $(z_k - \xi)$ norm converges to 0, we have

$$\{e_k, z_k, e_k\} - \{e_k, \xi, e_k\} = \{e_k, (z_k - \xi), e_k\} \quad \xrightarrow{w} \quad 0$$

by Lemma 4.3.2. Hence $\{e_k, z_k, e_k\} \xrightarrow{w} \{e, \xi, e\}$ by Lemma 4.3.1. Similarly $\{v_k, z_k, v_k\} \xrightarrow{w} \{v, \xi, v\}$. □

*Remark* 4.3.4. An examination of the proof of Lemma 4.3.2 reveals that for any $x \in L(\mathbb{C}^2, H)$ we also have $\{e_k, w_k, x\} \xrightarrow{w} 0$ and, akin to Corollary 4.3.3, we also have $\{e_k, z_k, x\} \xrightarrow{w} \{e, \xi, x\}$.

**Lemma 4.3.5.** *We have* $e = \{e, e, e\} + \beta\{e, v, e\}$ *and* $\beta v = \{v, e, v\} + \beta\{v, v, v\}$.

*Proof.* By Corollary 4.3.3 and triple orthogonality of $e_k$ and $v_k$, we have

$$\begin{aligned}
\alpha_k e_k &= \alpha_k\{e_k, e_k, e_k\} + \beta_k\{e_k, v_k, e_k\} \\
&= \{e_k, z_k, e_k\} \\
&\xrightarrow{w} \{e, \xi, e\} \\
&= \{e, e, e\} + \beta\{e, v, e\}.
\end{aligned}$$

Together with $\alpha_k e_k \xrightarrow{w} e$, this gives $e = \{e, e, e\} + \beta\{e, v, e\}$.

By Corollary 4.3.3 and triple orthogonality of $e_k$ and $v_k$, we have

$$\begin{aligned}
\beta_k v_k &= \alpha_k\{v_k, e_k, v_k\} + \beta_k\{v_k, v_k, v_k\} \\
&= \{v_k, z_k, v_k\} \\
&\xrightarrow{w} \{v, \xi, v\} \\
&= \{v, e, v\} + \beta\{v, v, v\}.
\end{aligned}$$

Together with $\beta_k v_k \xrightarrow{w} \beta v$, this gives $\beta v = \{v, e, v\} + \beta\{v, v, v\}$.

□

**Lemma 4.3.6.** *We have* $\{e, v, e\} = 0$ *and* $\{v, e, v\} = 0$.

*Proof.* Recall $e = a_e \otimes b_e$ and $v = a_v \otimes b_v$. We show $\{e, v, e\} = 0$, the proof of $\{v, e, v\} = 0$ is similar. We have

$$\begin{aligned}
\langle ev^* e(\mu), h\rangle &= \langle e(\mu), b_v\rangle\langle a_v, a_e\rangle\langle b_e, h\rangle \\
&= \lim_k \langle e(\mu), b_{v_k}\rangle\langle a_{v_k}, a_{e_k}\rangle\langle b_{e_k}, h\rangle \\
&= \lim_k \langle e_k v_k^* e(\mu), h\rangle \\
&= 0
\end{aligned}$$

for all $\mu \in \mathbb{C}^2$ and $h \in H$. Therefore we have $ev^*e = 0$. □

**Lemma 4.3.7.** *$e$ is a minimal tripotent and, if $\beta \neq 0$, $v$ is also a minimal tripotent.*

*Proof.* By Lemma 4.3.6, we have

$$e = \{e, e, e\} + \beta\{e, v, e\} = \{e, e, e\}$$

and

$$\beta v = \{v, e, v\} + \beta\{v, v, v\} = \beta\{v, v, v\}.$$

Let $\beta \neq 0$. Then $v = \{v, v, v\}$.

We show that $e \neq 0$. Assume $e = 0$, then $\beta v = \xi$ implies $\beta = 1$ and $\|v\| = 1$ which gives $1 = \|v\| = \sup_{\|\lambda\|=1} \|v(\lambda)\| = \sup_{\|\lambda\|=1} \|\langle\lambda, a_v\rangle b_v\| \leq \|b_v\| \leq 1$ and thus $\|b_v\| = 1$ and we have $v_k \to v$ in norm. This implies $e_k = \frac{1}{\alpha_k}(\alpha_k e_k + \beta_k v_k) - \frac{\beta_k}{\alpha_k} v_k \to e$ in norm, which is a contradiction as $\|e_k\| = 1$ for all $k$. Hence $e$ is a nonzero tripotent and it follows from $1 = \|e\| = \|a_e \otimes b_e\| = \|a_e\|\|b_e\|$ that $\|a_e\| = \|b_e\| = 1$. Therefore the sequence $(b_{e_k})$ norm converges to $b_e$ in the Hilbert space $H$, and $e$ is a minimal tripotent.

Let $\beta \neq 0$. We now show that $v$ is not zero. Assume $v = 0$, then $\xi = e$ and $e_k = a_{e_k} \otimes b_{e_k} \to a_e \otimes b_e = e$ in norm by the preceding paragraph (see Lemma

4.3.8 below). This implies $v_k = \frac{1}{\beta_k}(\alpha_k e_k + \beta_k v_k) - \frac{\alpha_k}{\beta_k} e_k \to \frac{1}{\beta}(\xi - e) = 0$ in norm, which is a contradiction as $\|v_k\| = 1$ for all $k$. Hence $v$ is a nonzero tripotent and, similarly to above, it follows from $1 = \|v\| = \|a_v \otimes b_v\| = \|a_v\|\|b_v\|$ that $\|a_v\| = \|b_v\| = 1$. Therefore the sequence $(b_{v_k})$ norm converges to $b_v$ in the Hilbert space $H$, and $v$ is a minimal tripotent. □

**Lemma 4.3.8.** *$(e_k)$ converges to $e$ in norm. If $\beta \neq 0$ then $(v_k)$ converges to $v$ in norm.*

*Proof.* We have already shown that $(b_{e_k})$ converges to $b_e$ in norm (see proof of Lemma 4.3.7). Hence

$$\begin{aligned}
\|e_k - e\| &= \|a_{e_k} \otimes b_{e_k} - a_e \otimes b_e\| \\
&\leq \|a_{e_k} \otimes b_{e_k} - a_e \otimes b_{e_k}\| + \|a_e \otimes b_{e_k} - a_e \otimes b_e\| \\
&= \|(a_{e_k} - a_e) \otimes b_{e_k}\| + \|a_e \otimes (b_{e_k} - b_e)\| \\
&= \|a_{e_k} - a_e\|_{\mathbb{C}^2} + \|b_{e_k} - b_e\|_H \\
&\to 0.
\end{aligned}$$

If $\beta \neq 0$, then the sequence $(b_{v_k})$ norm converges to $b_v$ (see proof of Lemma 4.3.7), and likewise $v_k = a_{v_k} \otimes b_{v_k} \to a_v \otimes b_v = v$ in norm. □

**Corollary 4.3.9.** *$e$ and $v$ are triple orthogonal.*

*Proof.* For triple orthogonality we must show $e \,\square\, v = 0$. The equations

$$\begin{aligned}
2(e \,\square\, v)(x)(z) &= ev^* x(z) + xv^* e(z) \\
&= \langle x(z), b_v\rangle\langle a_v, a_e\rangle b_e + \langle z, a_e\rangle\langle b_e, b_v\rangle x(a_v)
\end{aligned}$$

for all $x \in L(\mathbb{C}^2, H)$ and $z \in \mathbb{C}^2$ show that $e$ and $v$ are triple orthogonal if $\langle a_e, a_v\rangle = 0$ and $\langle b_e, b_v\rangle = 0$. As $a_e$ and $a_v$ are the norm limits of the $\mathbb{C}^2$ sequences $(a_{e_k})$ and $(a_{v_k})$ with $\langle a_{e_k}, a_{v_k}\rangle = 0$ for all $k \in \mathbb{N}$ we have $\langle a_e, a_v\rangle = 0$. The equation $\langle b_e, b_v\rangle = 0$ holds because, as noted in the proof of Lemma 4.3.7, the sequence $(b_{e_k})$ norm converges to $b_e$ in the Hilbert space $H$, which when coupled with the fact that the sequence $(b_{v_k})$ weakly converges to $b_v$ in $H$, gives the following standard calculation:

$$\begin{aligned} |\langle b_e, b_v\rangle| = |\langle b_{e_k}, b_{v_k}\rangle - \langle b_e, b_v\rangle| &= |\langle b_{e_k}, b_{v_k}\rangle - \langle b_e, b_{v_k}\rangle + \langle b_e, b_{v_k}\rangle - \langle b_e, b_v\rangle| \\ &\leq |\langle b_{e_k} - b_e, b_{v_k}\rangle| + |\langle b_e, b_{v_k} - b_v\rangle| \\ &\leq \|b_{e_k} - b_e\|\|b_{v_k}\| + |\langle b_e, b_{v_k} - b_v\rangle| \\ &\to 0. \end{aligned}$$ □

To summarise, we have the following dichotomy.

**Proposition 4.3.10.** *Exactly one of the following holds:*

*(i)* $\beta \neq 0$, *in which case* $e$ *and* $v$ *are triple orthogonal minimal tripotents, or*

*(ii)* $\beta = 0$, *in which case* $e$ *is a minimal tripotent and* $v$ *is triple orthogonal to* $e$.

In view of the above dichotomy, there will be no confusion in representing the Wolff point $\xi$ by $e + \beta v$ where $e$ and $v$ are mutually orthogonal minimal tripotents, and $\beta \in [0, 1]$.

We now determine the convergence properties of the basic Jordan operators induced by the sequences of tripotents $(e_k)$ and $(v_k)$. For this, we first derive a more general lemma.

**Lemma 4.3.11.** *If* $(T_k)$ *norm converges to* $T$ *in* $V$ *and* $(S_k)$ *weakly converges to* $S$ *in* $V$, *then the sequence* $(\{T_k, T_k, S_k\})$ *weakly converges to* $\{T, T, S\}$.

*Proof.* First we show that $(S_k T_k^* T_k)$ weakly converges to $ST^*T$. Let $\mu \in \mathbb{C}^2$ and $h \in H$.

We have

$$\begin{aligned}\langle S_k T_k^* T_k(\mu), h\rangle &= \langle T_k^* T_k(\mu), S_k^*(h)\rangle \\ &\to \langle T^* T(\mu), S^*(h)\rangle \\ &= \langle S T^* T(\mu), h\rangle,\end{aligned}$$

since $(T_k^* T_k(\mu))$ norm converges to $T^* T(\mu)$ and $(S_k^*(h))$ weakly converges to $S^*(h)$.

Likewise

$$\begin{aligned}\langle T_k T_k^* S_k(\mu), h\rangle &= \langle S_k(\mu), T_k T_k^*(h)\rangle \\ &\to \langle S(\mu), T T^*(h)\rangle \\ &= \langle T T^* S(\mu), h\rangle,\end{aligned}$$

as $(T_k T_k^*(h))$ norm converges to $T^* T(h)$ and $(S_k(\mu))$ weakly converges to $S(\mu)$.

$\square$

**Corollary 4.3.12.** *Let $(w_k)$ be a sequence in $\overline{D}$ converging weakly to $w \in \overline{D}$. Then $(e_k \square e_k)(w_k) \xrightarrow{w} (e \square e)(w)$.*

**Lemma 4.3.13.** *Let $(w_k)$ be a sequence in $\overline{D}$ converging weakly to $w \in \overline{D}$. Then $P_2(e_k)(w_k) = Q_{e_k}^2(w_k) \xrightarrow{w} Q_e^2(w) = P_2(e)(w)$.*

*Proof.* Observe that

$$\begin{aligned}\langle e_k e_k^* w_k e_k^* e_k(\lambda), h\rangle &= \langle w_k e_k^* e_k(\lambda), e_k e_k^*(h)\rangle \\ &\to \langle w e^* e(\lambda), e e^*(h)\rangle \\ &= \langle e e^* w e^* e(\lambda), h\rangle\end{aligned}$$

because $w_k e_k^* e_k(\lambda) \xrightarrow{w} w e^* e(\lambda)$ and $e_k e_k^*(h) \to e e^*(h)$. $\square$

**Lemma 4.3.14.** *Let $(w_k)$ be a sequence in $\overline{D}$ converging weakly to $w \in \overline{D}$. Then $P_0(e_k)(w_k) = B(e_k, e_k)(w_k) \xrightarrow{w} B(e, e)(w) = P_0(e)(w)$.*

*Proof.* In view of Corollary 4.3.12 and Lemma 4.3.13, the result follows from

$$\begin{aligned} B(e_k, e_k)(w_k) &= w_k - 2(e_k \,\square\, e_k)(w_k) + Q_{e_k}^2(w_k) \\ &\xrightarrow{w} w - 2(e \,\square\, e)(w) + Q_e^2(w) \\ &= B(e, e)(w). \end{aligned}$$

□

*Remark* 4.3.15. If $\beta \neq 0$, then Corollary 4.3.12 and Lemmas 4.3.13 and 4.3.14 hold when $e_k$ and $e$ are replaced by $v_k$ and $v$ respectively.

**Lemma 4.3.16.** *Let $(w_k)$ be a sequence in $\overline{D}$ converging weakly to $w \in \overline{D}$. If $\beta \neq 0$, then $(e_k \,\square\, e_k)(v_k \,\square\, v_k)(w_k) \xrightarrow{w} (e \,\square\, e)(v \,\square\, v)(w)$.*

*Proof.* If $\beta \neq 0$, then $(e_k)$ norm converges to $e$ and $(v_k)$ norm converges to $v$ by Corollary 4.3.8. The result follow from two applications of Lemma 4.3.11. □

*Remark* 4.3.17. Let $(w_k)$ be a sequence in $\overline{D}$ which weakly converges to $w \in \overline{D}$. Corollary 4.3.12 and Lemmas 4.3.13 and 4.3.14 guarantee that $P_i(e_k)(w_k) \xrightarrow{w} P_i(e)(w)$ for $i = 0, 1, 2$. If $\beta \neq 0$, then Remark 4.3.15 ensures we also have $P_i(v_k)(w_k) \xrightarrow{w} P_i(v)(w)$ for $i = 0, 1, 2$. In this case, we also have $P_{ij}(v_k, e_k)(w_k) \xrightarrow{w} P_{ij}(v, e)(w)$ for $0 \leq i \leq j \leq 2$.

Let $x \in \overline{D}$. As the triple product is norm continuous we have $P_i(e_k)(x) \to P_i(e)(x)$ for $i = 0, 1, 2$. If $\beta \neq 0$, we also have $P_i(v_k)(x) \to P_i(v)(x)$ for $i = 0, 1, 2$. In this case, we also have $P_{ij}(v_k, e_k)(x) \to P_{ij}(v, e)(x)$ for $0 \leq i \leq j \leq 2$.

## 4.4 Invariant domains

In this section, we will describe the invariant domains of a compact fixed-point-free holomorphic map $f : D \to D$ on the open unit ball $D$ of $V = L(\mathbb{C}^2, H)$.

Let $(z_k)$ be the sequence converging to the Wolff point $\xi = e + \beta v \in \partial D$ obtained in Section 4.3. We are going to construct a family of $f$-invariant domains at $\xi$, called horospheres, parameterised by positive real numbers. The construction is analogous to that given in Section 3.2 (*cf.* Remark 3.2.6). We recall the detail for convenience and completeness. For each $\lambda > 0$, we can find a sequence $(r_k)$ in $(0, 1)$ such that

$$\lambda = \frac{1 - r_k^2}{1 - \|z_k\|^2},$$

from some $k$ onwards. Note that $(r_k)$ converges to 1. For $y \in V$, denote by $D(y, r) := \{z \in V : \|z - y\| < r\}$ the open ball of $V$ centred at $y$ of radius $r > 0$ and define the *Kobayashi balls* $D_k[\lambda]$ by

$$\begin{aligned} D_k[\lambda] &:= g_{z_k}(\overline{D}(0, r_k)) \\ &= \{z \in V : \|g_{-z_k}(z)\| \leq r_k\}. \end{aligned}$$

Let

$$S(\xi, \lambda) := \{x \in \overline{D} : x = \lim_k x_k \text{ and } x_k \in D_k[\lambda]\}.$$

Then $\xi \in S(\xi, \lambda)$. We call $S(\xi, \lambda) \cap D$ a *horosphere* at $\xi$ and, by Remark 3.2.6, it is invariant under $f$. For completeness, we include a proof of this assertion.

**Proposition 4.4.1.** *Let $f : D \to D$ be a compact holomorphic map without fixed point. Then $f(S(\xi, \lambda) \cap D) \subset S(\xi, \lambda) \cap D$.*

*Proof.* Let $x \in S(\xi, \lambda) \cap D$. Then $x = \lim_k x_k$ for some $x_k \in D_k[\lambda]$. Applying the Schwarz-Pick Lemma gives

$$\|g_{-z_k}(f_k(x_k))\| = \|g_{-f_k(z_k)}(f_k(x_k))\| \leq \|g_{-z_k}(x_k)\| \leq r_k,$$

which implies $f_k(x_k) \in D_k[\lambda]$. We complete the proof by noting that $f(x) = f(\lim_k x_k) = \lim_k f(x_k) = \lim_k f_k(x_k) \in S(\xi, \lambda)$. $\square$

We will need a more explicit description of $S(\xi, \lambda)$ for future applications. Analogous to (3.2.2) and (3.2.3), we have

$$g_{z_k}(D(0, r_k)) = c_k(\lambda) + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(D),$$

and

$$D_k[\lambda] = g_{z_k}(\overline{D}(0, r_k)) = c_k(\lambda) + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(\overline{D}),$$

where $c_k(\lambda) = (1 - r_k^2) B(r_k z_k, r_k z_k)^{-1/2}(z_k)$.

Let $z_k = \alpha_k e_k + \beta_k v_k$, as in Section 4.3, where $\|z_k\| = \alpha_k$, $(e_k)$ norm converges to $e$, and $\beta \neq 0$ implies $(v_k)$ norm converges to $v$. Let $P_{ij}^k$ denote the joint Peirce projections $P_{ij}(v_k, e_k)$ for $0 \leq i \leq j \leq 2$, discussed at the end of Section 2.2. By (2.1.5) with $\lambda_{0k} = 0, \lambda_{1k} = r_k \beta_k$ and $\lambda_{2k} = r_k \alpha_k$, we have

$$\begin{aligned}
(1 - r_k^2) B(r_k z_k, r_k z_k)^{-1/2}(z_k) &= (1 - r_k^2) \sum_{0 \leq i \leq j \leq 2} (1 - \lambda_{ik}^2)^{-1/2} (1 - \lambda_{jk}^2)^{-1/2} P_{ij}^k(z_k) \\
&= \frac{\alpha_k (1 - r_k^2)}{1 - r_k^2 \alpha_k^2} e_k + \frac{\beta_k (1 - r_k^2)}{1 - r_k^2 \beta_k^2} v_k.
\end{aligned}$$

We also have

$$\begin{aligned}
\limsup_{k \to \infty} \alpha_k \frac{1 - r_k^2}{1 - r_k^2 \alpha_k^2} &= \limsup_{k \to \infty} \frac{1 - r_k^2}{1 - r_k^2 \|z_k\|^2} \\
&= \limsup_{k \to \infty} \frac{\frac{1 - r_k^2}{1 - \|z_k\|^2}}{1 + \frac{1 - r_k^2}{1 - \|z_k\|^2} \|z_k\|^2} \\
&= \limsup_{k \to \infty} \frac{\lambda}{1 + \lambda \|z_k\|^2} \\
&= \frac{\lambda}{\lambda + 1}
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{k\to\infty} \beta_k \frac{1-r_k^2}{1-r_k^2\beta_k^2} &= \beta \limsup_{k\to\infty} \frac{1-r_k^2}{1-r_k^2\beta_k^2} \\
&= \beta \limsup_{k\to\infty} \frac{\frac{1-r_k^2}{1-\|z_k\|^2}}{\frac{1-r_k^2}{1-\|z_k\|^2} + \frac{1-\beta_k{}^2}{1-\|z_k\|^2} r_k^2} \\
&= \beta \frac{\sigma\lambda}{\sigma\lambda+1},
\end{aligned}$$

where $0 \leq \beta \leq 1$ and

$$\sigma = \begin{cases} \frac{1}{\gamma} & \text{if } \gamma < \infty \\ 0 & \text{if } \gamma = \infty \end{cases}, \tag{4.4.1}$$

where $\gamma := \liminf_{k\to\infty} \frac{1-\beta_k{}^2}{1-\|z_k\|^2}$. Note that $\gamma \geq 1$ and hence $\sigma \in [0,1]$. Also $\beta = \lim_k \beta_k < 1$ implies $\sigma = 0$.

Hence

$$\limsup_{k\to\infty} \beta_k \frac{1-r_k^2}{1-r_k^2\beta_k^2} = \frac{\sigma\lambda}{\sigma\lambda+1}.$$

Choosing subsequences, we may assume the convergence of the following two sequences:

$$\frac{1-r_k^2}{1-r_k^2\alpha_k^2} = \frac{1-r_k^2}{1-r_k^2\|z_k\|^2} \to \frac{\lambda}{\lambda+1} \tag{4.4.2}$$

and

$$\frac{1-r_k^2}{1-r_k^2\beta_k^2} \to \frac{\sigma\lambda}{\sigma\lambda+1}. \tag{4.4.3}$$

Since $(e_k)$ norm converges to $e$, and $\beta \neq 0$ implies $(v_k)$ norm converges to $v$, we have the following norm convergence:

$$c_k(\lambda) = (1-r_k^2)B(r_kz_k, r_kz_k)^{-1/2}(z_k) \longrightarrow \frac{\lambda}{\lambda+1}e + \frac{\sigma\lambda}{\sigma\lambda+1}v. \tag{4.4.4}$$

We now give an explicit description of $S(\xi,\lambda)$ in terms of the Bergmann operator.

**Theorem 4.4.2.** *Let $f : D \to D$ be a compact holomorphic fixed-point-free self-map of the open unit ball $D$ of $L(\mathbb{C}^2, H)$. Then there exists a sequence $(z_k)$ in $D$ converging to a boundary point $\xi = e + \beta v$, where $\beta \in [0,1]$ and both $e$ and $v$ are mutually triple orthogonal minimal tripotents, such that for each $\lambda > 0$, the horosphere $S(\xi,\lambda)$ has the form*

$$\begin{aligned} S(\xi,\lambda) &= \frac{\lambda}{\lambda+1}e + \frac{\sigma\lambda}{\sigma\lambda+1}v \\ &\quad + B\left(\sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v, \sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v\right)^{1/2}(\overline{D}), \end{aligned}$$

*where $\sigma \in [0,1]$.*

*Proof.* Let $x \in S(\xi,\lambda)$ and let $(z_k)$ be the sequence from Section 4.3 converging to the Wolff point $\xi = e + \beta v$, where $z_k = \alpha_k e_k + \beta_k v_k$. Then there exist $x_k \in D_k[\lambda]$ such that $x = \lim_k x_k$, where

$$x_k = c_k(\lambda) + r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(w_k)$$

for some $w_k \in \overline{D}$. By weak compactness and choosing a subsequence, if necessary, we may assume $(w_k)$ has a weak limit $w \in \overline{D}$. By the equations for the square roots of the Bergmann operator (2.1.4) and (2.1.5), for any $T \in V$ we have

$$\begin{aligned} & r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(T) \\ =\ & r_k B(z_k, z_k)^{1/2} \sum_{0\le i\le j\le 2} (1 - r_k^2\lambda_{ik}^2)^{-1/2}(1 - r_k^2\lambda_{jk}^2)^{-1/2} P_{ij}(v_k, e_k)(T) \\ =\ & r_k \sum_{0\le i\le j\le 2} \sqrt{\frac{1-\lambda_{ik}^2}{1-r_k^2\lambda_{ik}^2}} \sqrt{\frac{1-\lambda_{jk}^2}{1-r_k^2\lambda_{jk}^2}} P_{ij}(v_k, e_k)(T) \qquad (4.4.5) \\ =\ & r_k \sum_{0\le i\le j\le 2} \sqrt{\frac{1}{r_k^2}\left(1 - \frac{1-r_k^2}{1-r_k^2\lambda_{ik}^2}\right)} \sqrt{\frac{1}{r_k^2}\left(1 - \frac{1-r_k^2}{1-r_k^2\lambda_{jk}^2}\right)} P_{ij}(v_k, e_k)(T), \qquad (4.4.6) \end{aligned}$$

where $\lambda_{0k} = 0, \lambda_{1k} = \beta_k$ and $\lambda_{2k} = \alpha_k$.

We proceed by considering two cases: $\beta = 1$ and $\beta \neq 1$. Let $\sigma \in [0,1]$ be as defined in (4.4.1). Note that $\sigma = 0$ if $\beta \neq 1$ whereas $\sigma \in [0,1]$ if $\beta = 1$.

*Case* 1. Suppose $\beta = 1$. Then, by Remark 4.3.17 along with (4.4.2), (4.4.3) and (4.4.6) , we have

$$
\begin{aligned}
& r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(w_k) \\
= \ & r_k \sum_{0 \le i \le j \le 2} \sqrt{\frac{1}{r_k^2}\left(1 - \frac{1 - r_k^2}{1 - r_k^2 \lambda_{ik}^2}\right)} \sqrt{\frac{1}{r_k^2}\left(1 - \frac{1 - r_k^2}{1 - r_k^2 \lambda_{jk}^2}\right)} P_{ij}(v_k, e_k)(w_k) \\
\overset{w}{\to} \ & \sum_{0 \le i \le j \le 2} \sqrt{1 - \delta_i^2}\sqrt{1 - \delta_j^2} P_{ij}(v, e)(w) \\
= \ & B(\delta_2 e + \delta_1 v, \delta_2 e + \delta_1 v)^{1/2}(w)
\end{aligned}
$$

where $\delta_i = \begin{cases} 0 & \text{if } i = 0 \\ \sqrt{\frac{\sigma\lambda}{\sigma\lambda + 1}} & \text{if } i = 1 \cdot \\ \sqrt{\frac{\lambda}{\lambda+1}} & \text{if } i = 2 \end{cases}$

Case 2. Suppose $\beta \neq 1$. Then $\left(\frac{1-\beta_k^2}{1-r_k^2\beta_k^2}\right)$ converges to 1 as $k \to \infty$. Note that, in this case, $\sigma = 0$. By Remark 4.3.17 along with (4.4.2) and (4.4.5),

$$
\begin{aligned}
& r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(w_k) \\
= \ & r_k \sum_{0 \le i \le j \le 2} \sqrt{\frac{1 - \lambda_{ik}^2}{1 - r_k^2 \lambda_{ik}^2}} \sqrt{\frac{1 - \lambda_{jk}^2}{1 - r_k^2 \lambda_{jk}^2}} P_{ij}(v_k, e_k)(w_k) \\
= \ & r_k \Bigg[ P_0(e_k)(w_k) - \left(1 - \sqrt{\frac{1 - \beta_k^2}{1 - r_k^2 \beta_k^2}}\right) P_{01}(v_k, e_k)(w_k) \\
& \qquad - \left(1 - \frac{1 - \beta_k^2}{1 - r_k^2 \beta_k^2}\right) P_{11}(v_k, e_k)(w_k) \Bigg] \\
& + \sqrt{1 - \frac{1 - r_k^2}{1 - r_k^2 \|z_k\|^2}} \Bigg[ P_1(e_k)(w_k) \\
& \qquad - \left(1 - \sqrt{\frac{1 - \beta_k^2}{1 - r_k^2 \beta_k^2}}\right) P_{12}(v_k, e_k)(w_k) \Bigg] \\
& + \frac{1}{r_k}\left(1 - \frac{1 - r_k^2}{1 - r_k^2 \|z_k\|^2}\right) P_2(e_k)(w_k)
\end{aligned}
$$

$$\begin{aligned}
&\overset{w}{\to} \quad P_0(e)(w) + \sqrt{1-\frac{\lambda}{\lambda+1}}P_1(e)(w) + \left(1-\frac{\lambda}{\lambda+1}\right)P_2(e)(w) \\
&= \quad B\left(\sqrt{\frac{\lambda}{\lambda+1}}e, \sqrt{\frac{\lambda}{\lambda+1}}e\right)^{1/2}(w),
\end{aligned}$$

where $\|P_{01}(v_k,e_k)\|, \|P_{11}(v_k,e_k)\|, \|P_{12}(v_k,e_k)\| \leq 1$ and we have implicitly used the fact that $P_0(e_k) = P_{01}(v_k,e_k) + P_{11}(v_k,e_k)$ and $P_1(e_k) = P_{02}(v_k,e_k) + P_{12}(v_k,e_k)$.

Therefore, in both cases, we have by (4.4.4),

$$\begin{aligned}
x &= \text{weak-}\lim_k x_k \\
&= \frac{\lambda}{\lambda+1}e + \frac{\sigma\lambda}{\sigma\lambda+1}v \\
&\quad + B\left(\sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v, \sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v\right)^{1/2}(w).
\end{aligned}$$

Conversely, let

$$y \in \frac{\lambda}{\lambda+1}e + \frac{\sigma\lambda}{\sigma\lambda+1}v + B\left(\sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v, \sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v\right)^{1/2}(\overline{D}).$$

Then

$$\begin{aligned}
y &= \frac{\lambda}{\lambda+1}e + \frac{\sigma\lambda}{\sigma\lambda+1}v \\
&\quad + B\left(\sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v, \sqrt{\frac{\lambda}{\lambda+1}}e + \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v\right)^{1/2}(x)
\end{aligned}$$

for some $x \in \overline{D}$. We show $y \in S(\xi, \lambda)$.

Let $y_k = c_k(\lambda) + r_k B(z_k,z_k)^{1/2}B(r_k z_k, r_k z_k)^{-1/2}(x)$. Then $y_k \in D_k[\lambda]$. We once again deal with two cases.

Case 1. Suppose $\beta = 1$. By Remark 4.3.17, along with (4.4.2), (4.4.3) and (4.4.6), we have the norm limit

$$\begin{aligned}
& r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(x) \\
= \; & r_k \sum_{0 \le i \le j \le 2} \sqrt{\frac{1}{r_k^2}\left(1 - \frac{1-r_k^2}{1-r_k^2\lambda_{ik}^2}\right)} \sqrt{\frac{1}{r_k^2}\left(1 - \frac{1-r_k^2}{1-r_k^2\lambda_{jk}^2}\right)} P_{ij}(v_k, e_k)(x) \\
\to \; & \sum_{0 \le i \le j \le 2} \sqrt{1-\delta_i^2}\sqrt{1-\delta_j^2} P_{ij}(v, e)(x) \\
= \; & B(\delta_2 e + \delta_1 v, \delta_2 e + \delta_1 v)^{1/2}(x),
\end{aligned}$$

where $\delta_i = \begin{cases} 0 & \text{if } i = 0 \\ \sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}} & \text{if } i = 1 \cdot \\ \sqrt{\frac{\lambda}{\lambda+1}} & \text{if } i = 2 \end{cases}$

From (4.4.4), we deduce that $y = \lim_k y_k \in S(\xi, \lambda)$.

Case 2. Suppose $\beta \neq 1$. Then $\left(\frac{1-\beta_k^2}{1-r_k^2\beta_k^2}\right)$ converges to 1 as $k \to \infty$. By Remark 4.3.17, along with (4.4.2) and (4.4.5), we have the following norm limit

$$\begin{aligned}
& r_k B(z_k, z_k)^{1/2} B(r_k z_k, r_k z_k)^{-1/2}(x) \\
= \; & r_k \sum_{0 \le i \le j \le 2} \sqrt{\frac{1-\lambda_{ik}^2}{1-r_k^2\lambda_{ik}^2}} \sqrt{\frac{1-\lambda_{jk}^2}{1-r_k^2\lambda_{jk}^2}} P_{ij}(v_k, e_k)(x) \\
= \; & r_k \left[ P_0(e_k)(x) - \left(1 - \sqrt{\frac{1-\beta_k^2}{1-r_k^2\beta_k^2}}\right) P_{01}(v_k, e_k)(x) \right. \\
& \qquad \left. - \left(1 - \frac{1-\beta_k^2}{1-r_k^2\beta_k^2}\right) P_{11}(v_k, e_k)(x) \right] \\
& + \sqrt{1 - \frac{1-r_k^2}{1-r_k^2\|z_k\|^2}} \left[ P_1(e_k)(x) \right. \\
& \qquad \left. - \left(1 - \sqrt{\frac{1-\beta_k^2}{1-r_k^2\beta_k^2}}\right) P_{12}(v_k, e_k)(x) \right] \\
& + \frac{1}{r_k}\left(1 - \frac{1-r_k^2}{1-r_k^2\|z_k\|^2}\right) P_2(e_k)(x) \\
\to \; & P_0(e)(x) + \sqrt{1 - \frac{\lambda}{\lambda+1}} P_1(e)(x) + \left(1 - \frac{\lambda}{\lambda+1}\right) P_2(e)(x)
\end{aligned}$$

$$= \quad B\left(\sqrt{\frac{\lambda}{\lambda+1}}e, \sqrt{\frac{\lambda}{\lambda+1}}e\right)^{1/2} (x),$$

where $\|P_{01}(v_k, e_k)\|\|P_{11}(v_k, e_k)\|, \|P_{12}(v_k, e_k)\| \leq 1$ and we have implicitly used the fact that $P_0(e_k) = P_{01}(v_k, e_k) + P_{11}(v_k, e_k)$ and $P_1(e_k) = P_{02}(v_k, e_k) + P_{12}(v_k, e_k)$.

Likewise (4.4.4) implies that $y = \lim_k y_k \in S(\xi, \lambda)$. □

## 4.5 Limit functions of iterates of holomorphic maps

Let $f$ be a self-map on a domain $D$ in a complex Banach space. Recall that a map $h : D \longrightarrow \overline{D}$ is called a *limit function* of the iterates $(f^n)$ if there is a subsequence $(f^{n_k})$ of $(f^n)$ converging to $h$ locally uniformly.

We make use of the following theorem which has been proved in [31, Theorem 3.1].

**Theorem 4.5.1.** *Let $B$ be the open unit ball in a Banach space and let $f : B \longrightarrow B$ be a compact map which is nonexpansive with respect to the Kobayashi distance. Then $f$ has a fixed point in $B$ if and only if there exists $z \in B$ and a subsequence of its iterates $(f^{n_k}(z))$ such that* $\sup_k \|f^{n_k}(z)\| < 1$.

For our purposes this theorem can be restated as follows:

**Theorem 4.5.2.** *Let $B$ be the open unit ball in a Banach space and let $f : B \longrightarrow B$ be a compact map which is nonexpansive with respect to the Kobayashi distance. Then $f$ is fixed-point-free if and only if for every $z \in B$ and every subsequence of its iterates $(f^{n_k}(z))$, we have* $\sup_k \|f^{n_k}(z)\| = 1$.

**Corollary 4.5.3.** *Let $B$ be the open unit ball in a Banach space and let $f : B \longrightarrow B$ be a fixed-point-free compact holomorphic map. Then $h(B) \subset \partial B$ for every limit function $h$ of $(f^n)$.*

*Proof.* Let $(f^{n_k})$ denote a subsequence of $(f^n)$ converging to $h$ uniformly on each open ball strictly contained in $B$. Choose a $z \in B$, then

$$\begin{aligned}
\|h(z)\| &= \|\lim_k f^{n_k}(z)\| \\
&= \lim_k \|f^{n_k}(z)\| \\
&= \limsup_k \|f^{n_k}(z)\| \\
&= \lim_{k\to\infty} (\sup_{\ell \geq k} \|f^{n_\ell}(z)\|) \\
&= 1,
\end{aligned}$$

where the last equality follows from Theorem 4.5.2. □

We will shortly describe the limit functions for a fixed-point-free compact holomorphic self-map $f$ on the open unit ball $D$ of $L(\mathbb{C}^2, H)$ with reference to boundary components.

As noted in [28], when $E$ is a finite rank JB*-triple with open unit ball $B$, every $a \in \overline{B}$ has a unique representation $a = e + v$, where $e$ is a tripotent and $v \in B_0(e) := B \cap E_0(e)$. The boundary component containing $a$ is given by $K_a = e + B_0(e)$, see Proposition 2.1.18. It is also important to note that every holomorphic map $h : D \longrightarrow \overline{D}$ has image in a single boundary component of $D$. For our rank-2 JB*triple $V = L(\mathbb{C}^2, H)$, the boundary components of $D$ in $\partial D$ take the form $K_e = e + D_0(e)$, where $e$ is a nonzero tripotent and $D_0(e) = D \cap V_0(e)$. In particular, $K_e = \{e\}$ when $e$ is a maximal tripotent. We now present our main theorem in this section.

**Theorem 4.5.4.** *Let $f : D \longrightarrow D$ be a fixed-point-free compact holomorphic self-map on the open unit ball $D$ of $L(\mathbb{C}^2, H)$. Then there exists a point $\xi = e + \beta v$ in the boundary $\partial D$, where $e$ and $v$ are mutually triple orthogonal minimal tripotents and $\beta \in [0, 1]$, such that for each limit function $h$ of $(f^n)$, we have $h(D) \subset K_u$ for some boundary component of a nonzero tripotent $u$ and either $\overline{K_u} \cap \overline{K_e}$ is a singleton or $u = e$.*

*Proof.* Employing Theorem 4.4.2, we construct a sequence from $\overline{h(D)}$. Choose a strictly increasing positive sequence $(\lambda_n)$ that tends to $\infty$. From each horosphere $S(\xi,\lambda_n)\cap D$ choose a point $y_n$. As $S(\xi,\lambda_n)\cap D$ is $f$-invariant and, by Corollary 4.5.3, $h(D)\subset\partial D$, we have $h(y_n)\in S(\xi,\lambda_n)\cap\partial D$ for each $n$. As noted above, the image $h(D)$ must lie in a single boundary component $K_u\subset\partial D$, where $u$ is either a minimal or a maximal tripotent. Let $h(y_n)=u+d_n$, where $d_n\in D_0(u)$ and, passing to a subsequence if necessary, we may assume $(d_n)$ weakly converges to $d\in\overline{D_0(u)}\subset V_0(u)$.

Recall that

$$
\begin{aligned}
S(\xi,\lambda) &= \frac{\lambda}{\lambda+1}e+\frac{\sigma\lambda}{\sigma\lambda+1}v \\
&\quad +B\left(\sqrt{\frac{\lambda}{\lambda+1}}e+\sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v,\sqrt{\frac{\lambda}{\lambda+1}}e+\sqrt{\frac{\sigma\lambda}{\sigma\lambda+1}}v\right)^{1/2}(\overline{D}),
\end{aligned}
$$

where $\sigma$ is defined as in (4.4.1).

We proceed by considering the cases $\sigma>0$ and $\sigma=0$ separately.

*Case* 1. Let $\sigma>0$, in which case $\beta=1$ and $\xi$ is a maximal tripotent which, of course, belongs to $\overline{K_e}$.

As $h(y_n)\in S(\xi,\lambda_n)\cap\partial D$, there exist $w_n\in\overline{D}$ such that

$$
\begin{aligned}
h(y_n) &= \lambda_{e,n}^2e+\lambda_{v,n}^2v+B(\lambda_{e,n}e+\lambda_{v,n}v,\lambda_{e,n}e+\lambda_{v,n}v)^{1/2}(w_n) \\
&= \lambda_{e,n}^2e+\lambda_{v,n}^2v \\
&\quad +(1-\lambda_{v,n}^2)^{1/2}P_{01}(v,e)(w_n)+(1-\lambda_{e,n}^2)^{1/2}P_{02}(v,e)(w_n) \\
&\quad +(1-\lambda_{e,n}^2)^{1/2}(1-\lambda_{v,n}^2)^{1/2}P_{12}(v,e)(w_n) \\
&\quad +(1-\lambda_{v,n}^2)P_{11}(v,e)(w_n)+(1-\lambda_{e,n}^2)P_{22}(v,e)(w_n),
\end{aligned}
$$

where $\lambda_{e,n}=\sqrt{\frac{\lambda_n}{\lambda_n+1}}$ and $\lambda_{v,n}=\sqrt{\frac{\sigma\lambda_n}{\sigma\lambda_n+1}}$.

Now $\lim_{n\to\infty}h(y_n)=e+v=\xi$. Therefore $\xi\in\overline{h(D)}\subset\overline{K_u}$.

*Case* 2. Let $\sigma = 0$.

First we note that weak-$\lim_n h(y_n) =$ weak-$\lim_n(u + d_n) = u + d \in u + \overline{D_0(u)} = \overline{K_u}$.

Now, as $h(y_n) \in S(\xi, \lambda_n) \cap \partial D$, there exist $w_n \in \overline{D}$ such that

$$\begin{aligned} h(y_n) &= \lambda_{e,n}^2 e + B(\lambda_{e,n} e, \lambda_{e,n} e)^{1/2}(w_n) \\ &= \lambda_{e,n}^2 e + P_0(e)(w_n) \\ &\quad +(1-\lambda_{e,n}^2)^{1/2} P_1(e)(w_n) + (1-\lambda_{e,n}^2) P_2(e)(w_n), \end{aligned} \tag{4.5.1}$$

where $\lambda_{e,n} = \sqrt{\frac{\lambda_n}{\lambda_n+1}}$.

By the weak compactness of $\overline{D}$, and passing to a subsequence if necessary, we may assume $(w_n)$ converges weakly to some $w \in \overline{D}$. Taking appropriate limits of (4.5.1) gives $u+d =$ weak-$\lim_{n\to\infty} h(y_n) = e + B(e,e)w \in \overline{K_e}$, by the weak continuity of $P_0(e)$. Hence we have $\emptyset \neq \overline{h(D)} \cap \overline{K_e} \subset \overline{K_u} \cap \overline{K_e}$.

We have shown that $\overline{K_u} \cap \overline{K_e} \neq \emptyset$ and Lemma 4.2.13 completes the proof. □

*Remark* 4.5.5. The following granular detail for Case 1 of the proof of Theorem 4.5.4 may be of interest. In this case, $\sigma$ defined as in (4.4.1) is positive and $\xi = e + v$ is a maximal tripotent in $\overline{K_u}$. On the one hand, if $u$ is a minimal tripotent then the maximal tripotent $\xi \notin h(D)$ because $h(D) \subset K_u$ and $\xi \notin K_u$, whereas, on the other hand, if $u$ is a maximal tripotent then $K_u = \{u\}$ and so $u = \xi$ and $h$ is the constant map taking value $\xi$.

# Appendix

**Proposition 1.1.11.** *The $E_n$ from Lemma 1.1.8 satisfy*

$$E_n = \sum_{r=0}^{n-1} \alpha^r \sum_{s=0}^{\min\{r,n-1-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} \quad (n \geq 1). \tag{1.1.2}$$

*Proof.* The left and right-hand side of (1.1.2) agree for $n = 1$. Assume that for some $n \in \mathbb{N}$ the formula holds for all $k \leq n$. Then

$$\begin{aligned}
E_{n+1} &= E_n + |a|^2 \sum_{r=1}^{n-1} E_{n-r}\alpha^r + \alpha^n \\
&= \sum_{r=0}^{n-1} \alpha^r \sum_{s=0}^{\min\{r,n-1-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} \\
&\quad + |a|^2 \sum_{r=1}^{n-1} \alpha^r \sum_{r_2=0}^{n-r-1} \alpha^{r_2} \sum_{s_2=0}^{\min\{r_2,n-r-1-r_2\}} |a|^{2s_2} \begin{pmatrix} r_2 \\ s_2 \end{pmatrix} \begin{pmatrix} n-r-1-r_2 \\ s_2 \end{pmatrix} + \alpha^n \\
&= 1 + \sum_{r=1}^{n-1} \alpha^r \left( \sum_{s=0}^{\min\{r,n-1-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} \right. \\
&\qquad \left. + \sum_{r_2=0}^{n-r-1} \alpha^{r_2} \sum_{s_2=0}^{\min\{r_2,n-r-1-r_2\}} |a|^{2(s_2+1)} \begin{pmatrix} r_2 \\ s_2 \end{pmatrix} \begin{pmatrix} n-r-1-r_2 \\ s_2 \end{pmatrix} \right) + \alpha^n \\
&= 1 + \sum_{r=1}^{n-1} \alpha^r \left( \sum_{s=0}^{\min\{r,n-1-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-1-r \\ s \end{pmatrix} \right. \\
&\qquad \left. + \sum_{s=1}^{r} \sum_{s_2=0}^{\min\{r-s,n-1-r\}} |a|^{2(s_2+1)} \begin{pmatrix} r-s \\ s_2 \end{pmatrix} \begin{pmatrix} n-1-r \\ s_2 \end{pmatrix} \right) + \alpha^n \quad \text{(by } (\star) \text{ below)} \\
&= 1 + \sum_{r=1}^{n-1} \alpha^r \left( \sum_{s_2=0}^{\min\{r,n-1-r\}} |a|^{2s_2} \begin{pmatrix} r \\ s_2 \end{pmatrix} \begin{pmatrix} n-1-r \\ s_2 \end{pmatrix} \right. \\
&\qquad \left. + \sum_{s=1}^{r} \sum_{s_2=1}^{\min\{r-s+1,n-r\}} |a|^{2s_2} \begin{pmatrix} r-s \\ s_2-1 \end{pmatrix} \begin{pmatrix} n-1-r \\ s_2-1 \end{pmatrix} \right) + \alpha^n \\
&= 1 + \alpha^n + \sum_{r=1}^{\lfloor \frac{n-1}{2} \rfloor} \alpha^r \left( \sum_{s_2=0}^{r} |a|^{2s_2} \begin{pmatrix} r \\ s_2 \end{pmatrix} \begin{pmatrix} n-1-r \\ s_2 \end{pmatrix} \right. \\
&\qquad \left. + \sum_{s=1}^{r} \sum_{s_2=1}^{r-s+1} |a|^{2s_2} \begin{pmatrix} r-s \\ s_2-1 \end{pmatrix} \begin{pmatrix} n-1-r \\ s_2-1 \end{pmatrix} \right)
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{r=\lfloor\frac{n-1}{2}\rfloor+1}^{n-1} \alpha^r \left( \sum_{s_2=0}^{n-1-r} |a|^{2s_2} \binom{r}{s_2} \binom{n-1-r}{s_2} \right. \\
& \qquad \left. + \left( \sum_{s=1}^{2r-(n-1)} \sum_{s_2=1}^{n-r} + \sum_{s=2r-(n-2)}^{r} \sum_{s_2=1}^{r-s+1} \right) |a|^{2s_2} \binom{r-s}{s_2-1} \binom{n-1-r}{s_2-1} \right) \\
= \;& 1+\alpha^n + \sum_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \alpha^r \left( \sum_{s_2=0}^{r} |a|^{2s_2} \binom{r}{s_2} \binom{n-1-r}{s_2} \right. \\
& \qquad \left. + \sum_{s_2=1}^{r} |a|^{2s_2} \binom{n-1-r}{s_2-1} \sum_{s=1}^{r-s_2+1} \binom{r-s}{s_2-1} \right) \qquad \text{(by } (\star\star) \text{ below)} \\
& + \sum_{r=\lfloor\frac{n-1}{2}\rfloor+1}^{n-1} \alpha^r \left( \sum_{s_2=0}^{n-1-r} |a|^{2s_2} \binom{r}{s_2} \binom{n-1-r}{s_2} \right. \\
& \qquad \left. + \left( \sum_{s_2=1}^{n-r} \sum_{s=1}^{2r-(n-1)} + \sum_{s_2=1}^{n-1-r} \sum_{s=2r-(n-2)}^{r+1-s_2} \right) |a|^{2s_2} \binom{r-s}{s_2-1} \binom{n-1-r}{s_2-1} \right) \\
& \qquad \text{(by } (\star\star\star) \text{ below)} \\
= \;& 1+\alpha^n + \sum_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \alpha^r \left( 1 + \sum_{s_2=1}^{r} |a|^{2s_2} \binom{r}{s_2} \left\{ \binom{n-1-r}{s_2} + \binom{n-1-r}{s_2-1} \right\} \right) \\
& + \sum_{r=\lfloor\frac{n-1}{2}\rfloor+1}^{n-1} \alpha^r \left( \sum_{s_2=0}^{n-1-r} |a|^{2s_2} \binom{r}{s_2} \binom{n-1-r}{s_2} \right. \\
& \qquad + \left( \sum_{s_2=1}^{n-r} |a|^{2s_2} \binom{n-1-r}{s_2-1} \sum_{s=1}^{2r-(n-1)} \binom{r-s}{s_2-1} \right. \\
& \qquad \left. \left. + \sum_{s_2=1}^{n-1-r} |a|^{2s_2} \binom{n-1-r}{s_2-1} \sum_{s=2r-(n-2)}^{r+1-s_2} \binom{r-s}{s_2-1} \right) \right) \\
= \;& 1+\alpha^n + \sum_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \alpha^r \sum_{s_2=0}^{r} |a|^{2s_2} \binom{r}{s_2} \binom{n-r}{s_2} \\
& + \sum_{r=\lfloor\frac{n-1}{2}\rfloor+1}^{n-1} \alpha^r \left( 1 + \sum_{s_2=1}^{n-1-r} |a|^{2s_2} \binom{r}{s_2} \binom{n-1-r}{s_2} + |a|^{2(n-r)} \sum_{s=1}^{2r-(n-1)} \binom{r-s}{n-r-1} \right. \\
& \qquad \left. + \sum_{s_2=1}^{n-1-r} \left( \binom{n-1-r}{s_2-1} \sum_{s=1}^{r+1-s_2} \binom{r-s}{s_2-1} \right) |a|^{2s_2} \right) \\
= \;& 1+\alpha^n + \sum_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \alpha^r \sum_{s_2=0}^{r} |a|^{2s_2} \binom{r}{s_2} \binom{n-r}{s_2} \\
& + \sum_{r=\lfloor\frac{n-1}{2}\rfloor+1}^{n-1} \alpha^r \left( 1 + \sum_{s_2=1}^{n-1-r} |a|^{2s_2} \binom{r}{s_2} \left\{ \binom{n-1-r}{s_2} + \binom{n-1-r}{s_2-1} \right\} \right. \\
& \qquad \left. + |a|^{2(n-r)} \binom{r}{n-r} \right) \\
= \;& 1+\alpha^n + \sum_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \alpha^r \sum_{s_2=0}^{r} |a|^{2s_2} \binom{r}{s_2} \binom{n-r}{s_2} \\
& + \sum_{r=\lfloor\frac{n-1}{2}\rfloor+1}^{n-1} \alpha^r \sum_{s_2=0}^{n-r} |a|^{2s_2} \binom{r}{s_2} \binom{n-r}{s_2} \\
= \;& 1 + \sum_{r=1}^{n-1} \alpha^r \sum_{s_2=0}^{\min\{r,n-r\}} |a|^{2s_2} \binom{r}{s_2} \binom{n-r}{s_2} + \alpha^n
\end{aligned}
$$

$$= \sum_{r=0}^{n} \alpha^r \sum_{s=0}^{\min\{r,n-r\}} |a|^{2s} \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} n-r \\ s \end{pmatrix},$$

where we have implicitly used the standard facts about binomial coefficients:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n-1 \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$$

$$\begin{pmatrix} n+1 \\ k+1 \end{pmatrix} = \sum_{i=k}^{n} \begin{pmatrix} i \\ k \end{pmatrix}$$

for $n \in \mathbb{N}$ and $1 \leq k \leq n$, and the double sum formulae:

$$\sum_{r=1}^{n-1} \alpha^r \sum_{r_2=0}^{n-r-1} \alpha^{r_2} C(r, r_2) = \sum_{r=1}^{n-1} \alpha^r \sum_{s=1}^{r} C(s, r-s) \qquad (\star)$$

$$\sum_{s=1}^{r} \sum_{s_2=1}^{r-s+1} C(s, s_2) = \sum_{s_2=1}^{r} \sum_{s=1}^{r-s_2+1} C(s, s_2) \qquad (\star\star)$$

$$\sum_{s=2r-(n-2)}^{r} \sum_{s_2=1}^{r-s+1} C(s, s_2) = \sum_{s_2=1}^{n-1-r} \sum_{s=2r-(n-2)}^{r-s_2+1} C(s, s_2) \qquad (\star\star\star)$$

where the $C(x, y)$ are coefficients dependent on $x, y \in \mathbb{N}$. $\square$

# List of Figures

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